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A STUDY OF THE EFFECTS OF NOISE JAMMING  
ON RADAR RECEIVERS

THESIS

Paul E. Bishop  
Major, USAF

AFIT/GE/ENG/90D-05

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A STUDY OF THE EFFECTS OF NOISE JAMMING  
ON RADAR RECEIVERS

THESIS

Presented to the Faculty of the School of Engineering  
of the Air Force Institute of Technology  
Air University  
In Partial Fulfillment of the  
Requirements for the Degree of  
Master of Science in Electrical Engineering

Paul E. Bishop, B.S.E.E.

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December, 1990

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## *Preface*

The goal of this study was to provide a single source document on the effects of noise jamming on pulsed radar receivers. This study would cover three main topics in depth. These areas to be covered are the sources of electrical noise, various noise jamming waveforms and the effects of each jamming waveform on various pulsed radar receivers.

This thesis effort is concentrated on determining the power spectra of various noise jamming waveforms, determining the probability densities of the outputs of conventional pulsed radar receivers and the effects of CW interference on conventional pulsed receivers. As such, this thesis is only the beginning of what will hopefully be a continuing effort here at AFIT. Although several texts were referenced in this effort, one document was extremely valuable. This important work was *Threshold Signals* by James L. Lawson and George E. Uhlenbeck.

I am deeply indebted to my thesis advisor, Dr. Vittal P. Pyati, for his help in preparing this thesis. He was able to keep me on a consistent track when I would begin to wander. I also wish to thank Lt Col Meer and Lt Col Norman who both provided valuable assistance toward this effort. They also sat on my thesis committee, for which I am indebted. I have to especially thank my wife, Sue, for allowing me the flexibility to work at all hours at home without interruption. It really made a difference.

Paul E. Bishop

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*Abstract*

A comprehensive study of the effects of noise jamming on various pulsed radar receiver types is proposed. First, a literature review on the sources of electrical noise and various jamming waveforms is presented. The power spectra for three noise jamming waveforms are rigorously derived. These cases are shot noise in a parallel-plate diode, a series of pulses with random amplitudes and intervals, and for a series of pulses with random amplitudes, spacing and phases. Next, the probability density functions for the output of linear and quadratic detectors are developed. For each detector type, the probability density functions for the cases of noise only and signal plus noise are derived. Finally, the effects of CW interference on conventional pulsed radar receivers is accomplished. This analysis shows that the effect of a pure CW tone as an interfering signal is that it increases the minimum power required for detection. As the CW tone is detuned from the signal carrier frequency the minimum power required for detection increases until the detuning is so great that the effects of the CW is negligible. This thesis lays the ground work for a much broader future study of the effects of noise jamming on various pulsed radar receiver types.

— *Thesis* — RRH

# A STUDY OF THE EFFECTS OF NOISE JAMMING ON RADAR RECEIVERS

## *I. Introduction*

A study of the effects of noise jamming on radar receivers is proposed. The result of this study will be a comprehensive reference on the effects of various types of noise jamming on different types of radar receivers.

### *1.1 Background*

*1.1.1 Radar Principles* Radar is an acronym for radio detection and ranging. A radar system can measure the range to a target by transmitting a pulse of electromagnetic energy, and measuring the time the pulse takes to reflect off a target and return to the radar receiver. The time for a pulse to travel one nautical mile and return is called a radar mile. A radar mile is defined as  $12.35\mu\text{sec}$  (6:8). However, before any information can be obtained from a transmitted radar signal, that signal must first be detected.

A radar receiver detects a target by comparing the voltage of the received signal with a preset threshold voltage. If the received voltage is greater than the threshold, the radar is said to have detected a target. Radar receivers are designed to meet two main opposing specifications. These are the probability of detection ( $P_d$ ) and probability of false alarm ( $P_{fa}$ ). A radar's probability of detection is a measure of how often the radar will actually detect a target when a target is present. The probability of false alarm is a measure of how often a radar will decide that a target is present when there is not a target present. The parameters of  $P_d$  and  $P_{fa}$  are used to determine a radar's required signal-to-noise ratio (SNR) and the minimum required signal power (18.28). An increase in the noise power will result in a decrease in the signal-to-noise ratio. The decrease in SNR, which

is due to an increase in noise power, causes a decrease in the radar's probability of detection and an increase in the radar's probability of false alarm.

*1.1.2 Radar Receiver Types* There are several types of radar receivers in use today. Four of the most common radar receiver designs can be grouped into four main categories. These are the conventional receiver, the matched filter receiver, the pulse compression receiver, and the constant false alarm rate (CFAR) receiver.

The conventional receiver uses a nonlinear element followed by a threshold comparison device to detect the presence of a radar pulse. One example of a conventional receiver is a simple envelope detector, also known as a linear detector. A second type of conventional receiver employs a quadratic detector. This type of detector is also referred to as a square law detector. A matched filter receiver is designed to maximize the output signal-to-noise ratio (18:369). The matched filter compares the received waveform with a stored replica of the transmitted pulse. One commonly used method of implementing the matched filter is with a correlator (18:375). A correlator multiplies a received signal with a reference signal, which is matched to the transmitted pulse, and then integrates the resulting product over a specified time period. The type of pulse compression receiver analyzed is the linear FM chirp receiver. This type of receiver is employed in high range resolution imaging radars. The CFAR receiver will be the last receiver structure analyzed. The CFAR receiver employs an adaptive threshold to maintain a constant probability of false alarm (18:39). These four receiver types are the most common structures found in search radar systems.

*1.1.3 Noise Principles* Electrical noise is electromagnetic energy across a broad range of frequencies, which is characterized by random fluctuations of amplitude, frequency, phase, etc. Noise is the chief parameter that limits a radar receiver's sensitivity (18:18). Noise is also the primary factor affecting a receiver's  $P_d$  and  $P_{fa}$ . There are several sources of electrical noise. Two main categories of noise are the noise that originates within the receiver and noise external to the receiver. Thermal noise is the primary type of noise generated in a radar receiver. The power of

thermal noise is directly proportional to the temperature of the receiver. Thermal noise is present to a certain degree in all electrical devices. Some types of external noise sources are man made noise, cosmic noise and electronic countermeasures (ECM). The effects of thermal noise can be reduced by proper receiver design, but can never be totally eliminated. Radar receivers are designed for a specific  $P_d$  and  $P_{fa}$  given the known value of thermal noise generated in the receiver.

A radar signal must travel through an environment where electrical noise is present. A radar receiver receives a signal that is the summation of the desired radar waveform and the external noise (15:36). A large value of noise power may cause the receiver to announce the presence of a target when in reality, no target is present. This false detection is termed a false alarm, and increases the radar's probability of false alarm. A second effect of the external noise is to force the receiver to conclude there is no target when an actual target is present, thereby decreasing the radar's probability of detection.

*1.1.4 Noise Jamming* Noise jamming is one of several active electronic countermeasures techniques. The primary noise jamming method is to produce a signal that is as random, or noise-like, as possible (6:83). The optimum noise jamming signal produced by a jammer should be a close approximation of the thermal noise that is produced by the radar receiver (20:293).

One must consider several variables when analyzing the noise jamming problem. The jammer power is one of the most critical parameters. To be effective, the level of the jamming must be greater than the energy of the received pulsed radar signal (6:89). The statistical characteristics of the noise jamming waveform are also important. For a noise jamming signal to be effective, the statistics of the noise should be as close to the statistics of the receiver thermal noise as possible. Similarly, a deception jammer attempts to mimic a radar target return. The primary goal of noise jamming is to increase the radar's  $P_{fa}$  and decrease the radar's  $P_d$  by producing a high power noise waveform that approximates the thermal noise already present in the receiver.

## *1.2 Problem*

Adequate employment of radar systems, in an environment where noise jamming is used by an enemy, requires knowledge of the effects of the noise jamming on the radar receiver. Conversely, effective noise jamming against an enemy radar requires the knowledge of the effects of the noise jamming on the enemy's radar receiver. Presently, no comprehensive study exists on the effects of various noise jamming waveforms on different types of radar receivers' probability of false alarm and probability of detection. A comprehensive study includes the characterization of the sources of electrical noise, the statistical parameters of different noise jamming waveforms, and an analysis of the effects of these jamming waveforms on various radar receiver types. The goal of this thesis effort is to provide such a comprehensive study. The result of this effort will be a single source of reference for the effects of noise jamming on radar receivers.

## *1.3 Assumptions*

Several assumptions must be made in the analysis of the effects of noise jamming on radar receivers. The effects of jammer antenna gain, target characteristics, range, and jammer power can be disregarded. Instead, the jammer-to-signal (J/S) ratio will be used in place of these parameters (1:14-11-14-12). By the same reasoning, propagation effects will be disregarded. The radar receiver will be assumed to have linear response characteristics up to the detector stage. This assumption is a reasonable approximation, which allows for much simpler calculations. However, the detector is generally a nonlinear network and cannot be modeled with linear characteristics. For all analyses, only a single pulse will be considered. For the multiple pulse case there are simple techniques available to determine the required signal-to-noise ratio. A lossless system will be assumed. Any system losses will be a constant factor that will not impact the effects of the noise jamming. Finally, the radar receiver will be assumed not to incorporate any specific electronic counter-countermeasures (ECCM). All these assumptions lead to a lossless receiver model that is linear up

to the detector stage, which is affected only by the receiver's thermal noise and the external noise jamming. Therefore, the impact on the modeled receiver's  $P_d$  and  $P_{fa}$  will be due solely to the radar jamming.

#### *1.4 Approach*

The study of the effects of noise jamming will be approached in three parts. First, the sources of electrical noise will be characterized. Next, the noise jamming waveform statistics will be modeled. Finally, the noise jamming waveform statistics will be applied to the four different receiver models to determine the effects of the jamming on the receiver's  $P_{fa}$  and  $P_d$ .

The sources of electrical noise will be characterized by their statistical properties, specifically by their probability density and power spectral density. The two primary sources of noise to be modeled are thermal noise and shot noise. Both of these sources can be modeled as white (uniform power spectral density) and Gaussian (normal) density. This model of noise most closely resembles the noise present in the receiver. Shot noise and thermal noise are the primary sources of noise used in radar noise jammers. These two noise sources are the most popular because they are simple to generate by radar jammers.

After the sources of noise have been modeled, the noise jamming waveforms must be characterized. As with the noise sources, the noise jamming waveforms are modeled by their statistical properties. Although the noise jamming waveforms under consideration approach the ideal receiver thermal noise characteristics, there are differences that must be considered in the analysis of their effects on the radar receivers. These differences are reflected in the noise power spectral density and probability density function. The noise jamming waveforms to be modeled are direct noise amplified (DINA), frequency modulated (FM) by-noise, and amplitude modulated (AM) by-noise. Also, continuous wave (single frequency) and random pulse waveforms must be considered. These waveforms are used extensively in current radar noise jammers.

The analysis of the jamming effects on the receivers will follow techniques demonstrated by Lawson and Uhlenbeck (11) and those outlined by Bennighof and others, as well as several other radar text books (1). The statistical characteristics of the noise jamming waveform will be applied to the transfer characteristic of the radar receiver to determine the output noise characteristics. The result of this analysis will be the probability density and power spectral density of the noise output of the receiver. This will be accomplished for the cases where an actual target is present and when a target is not present. One technique to analyze the effects of the radar jamming is to apply the detectability criteria outlined by Lawson and Uhlenbeck (11:161-165). The application of these criteria will give an indication to the required power levels for detection. A second technique is to apply the two probability densities to the known decision technique for each radar receiver type. This technique is similar to the likelihood ratio test. The results of the likelihood ratio will determine the regions where the  $P_d$  and  $P_{fa}$  can be calculated. The resulting  $P_d$  and  $P_{fa}$  will be compared with the  $P_d$  and  $P_{fa}$  for the receiver when no noise jamming is present. The difference in the receiver's  $P_d$  and  $P_{fa}$  will be due to the noise jamming. This analysis will be accomplished for each of the noise jamming waveforms through each of the separate receiver types.

### 1.5 Scope

This study is limited to the analysis of the effects of noise jamming on search radars. However, the techniques used are applicable to the analysis of the effects on tracking radars as well. Four types of noise jamming waveforms will be used in the analysis. These waveforms are continuous wave (CW) interference, DINA, AM by-noise and FM by-noise. These waveforms are the major jamming signals employed in current noise jammers (1.14-9). Only pulsed radars will be considered. With some slight differences, the methods employed in the analysis can be applied to other radar types, such as continuous wave (CW) and pulse doppler (PD) radars. Four basic pulsed radar receiver types will be analyzed in this study. These receiver types are the conventional receiver, the matched filter receiver, the pulse compression receiver and the CFAR receiver.

This thesis effort, as a part of a much larger study, is limited to the analysis of the effects of continuous wave interference on two types of conventional receiver detector structures: the linear detector, and the quadratic detector. The analysis is limited to the application of the deflection criterion outlined by Lawson and Uhlenbeck to determine the minimum signal power required for detection (11:161). Although this effort is limited to the analysis of CW interference on conventional receivers, the techniques developed here can be easily applied to the analysis of the effects on other receiver structures by various jamming waveforms.

### *1.6 Summary*

In this chapter, a study of the effects of noise jamming on various radar receivers was proposed. This chapter also provided the necessary background, assumptions, and the approach to accomplish this study. This chapter also outlined the scope of this particular thesis effort. Chapter II is a literature search on the sources of electrical noise and on various noise jamming waveforms. In Chapter III the power spectra for three specific jamming waveforms are derived. The probability density functions for the output of linear and quadratic detectors are developed in Chapter IV. Probability density functions for the cases of noise only and of signal plus noise are derived for each detector type. The effects of continuous wave interference on linear and quadratic detectors are analyzed in Chapter V. Lastly, Chapter VI contains conclusions and recommendations for further work in this study.



## *II. Literature Review*

Before analyzing the effects of noise jamming on radar receivers, one needs an understanding of two main subject areas. These subject areas are the sources of electrical noise used in radar jammers and noise jamming waveforms. The critical parameters to consider for the sources of noise are the signal's probability density function and power spectral density. These two parameters are also important to the study of noise jamming waveforms. A great deal of research has been conducted in these areas. Therefore, the goal of this literature review is to determine the most recent research in these areas, and the results of that research.

### *2.1 Noise Sources*

The basis of most radar jamming signals is a source of electrical noise. Any noise source may be generally categorized in one of two ways, either by its power spectral density (PSD) or the probability density function (PDF) of one of its parameters (usually the amplitude of the noise voltage). The PDF of a signal is a measure of how likely a particular amplitude of the signal is, whereas the PSD of a signal is a measure of how much power is contained in specific frequency bands. Turner and others point out that "...the ECM goal is to produce white Gaussian noise in the victim radar receiver" (19:118). Gaussian refers to the PDF of the amplitude of the noise. This type of noise has a Gaussian or normal amplitude PDF (represented by the familiar bell shaped curve). White noise is noise that has a uniform or flat PSD over all frequencies, analogous to the concept of white light, which contains light at all visible wavelengths (9:48). The true goal of noise jamming is to produce a Rayleigh distributed noise output from the victim receiver's detector (7:66). The Rayleigh PDF is closely related to the Gaussian PDF. Rayleigh distributed noise describes the envelope of the output of a radar's intermediate frequency (IF) filter when the input of the filter is Gaussian noise (18:24). It is sufficient for the analysis of the effects of noise on radar receivers that the PSD of the noise be uniform only over a specific frequency bandwidth of interest, generally the

receiver bandwidth. There are several sources of white Gaussian noise. The most common sources are thermal noise and shot noise.

Thermal noise is due to the random motion or fluctuations of electrons in conductors (9:47). When quantum mechanical effects are neglected, this type of noise has a flat or uniform power spectral density, directly proportional to the temperature of the device (21:9). By the central limit theorem, which states that for the sum of a large number of independent samples the limiting form of the PDF is Gaussian, thermal noise has a Gaussian PDF with a mean value of zero and a variance equal to the mean square value of the noise voltage (9:48-49). Thermal noise is present to a certain extent in all electrical devices and can be exploited as a noise source for radar jammers. Another source of electrical noise commonly used in radar jammers is shot noise.

Shot noise results from the flow of electrons in active devices such as vacuum tubes and semiconductors. In vacuum tubes, the shot noise is caused by the current induced from the random emission of electrons from the heated cathode (4:112). The shot noise in semiconductors is caused by current flows generated when carriers (electrons and holes) randomly crossing the p-n barrier (22:93). Davenport and Root provide the development of the PSD and PDF of shot noise in thermionic vacuum tubes for both the temperature-limited diode and the space-charge-limited diode cases (4:112-143). They showed that for frequencies less than the reciprocal of the electron transit time in the diode, the PSD of shot noise can be approximated by a flat spectrum (4:123). This spectrum is proportional to the average current through the diode, as opposed to the device temperature as is the case for thermal noise. Shot noise is also a phenomenon present in semiconductors. Van der Ziel developed expressions for shot noise in semiconductor diodes for both the low frequency and high frequency cases (22:93-109). He also presents an expression for the PSD of noise from a back-biased p-n diode, and states that large amounts of noise power can be generated in this case (22:48). Penney and others describe the back-biased diode as an "extremely hot noise generator" (9:71). By the application of the central limit theorem, the PDF of shot noise in both

vacuum tube diodes and semiconductor diodes can be shown to approach a Gaussian density. Shot noise, like thermal noise, is one excellent source of white Gaussian noise.

The properties of Gaussian noise make it an ideal choice for radar jamming signals. Turner and others define a measure of jamming performance, the noise quality (19:117-122). The noise quality is a measure of how closely the PDF of the jamming signal approaches that of a Gaussian density. The authors show that a high value of noise quality can lead to a reduction in the required jamming-to-signal ratio (J/S) (19:117). For a given level of effectiveness against a pulsed radar, this reduction in J/S can be as much as 17 dB (a factor of 50) over a low noise quality signal that produces the same effect (19:117). Based on the work done by Turner and others, Knorr and Karantanas outline a computer simulation, in use at the Naval Postgraduate School, for the optimization of various jamming waveforms (10:273-277). This computer simulation compares the PDF of the jamming waveform with a Gaussian PDF, having the same mean and variance, to arrive at a value for noise quality, as defined by Turner and others (10:274). Work continues at the Naval Postgraduate School in this area. Even though Gaussian noise is the most common source of noise for radar jammers, other noise densities can also be used. Two such noise sources are random pulses and single frequency, or continuous wave (CW) noise.

Random pulses can be used as one source of noise for a jamming signal. Lawson and Uhlenbeck present a thorough analysis of the power spectra of random pulse signals (11:43-46). They provide the derivation of the power spectra for a series of pulses with random amplitudes, random pulse repetition intervals, and random phases (11:43-46). The power spectrum for a series of pulses with random amplitudes is a continuous spectrum that has the same shape as the power spectrum of a single pulse (11:44). However, the power spectrum for a series of pulses with random pulse repetition intervals is dependent on both the spectrum of a single pulse and the probability density of the pulse intervals (11:44). Maksimov and others present expressions for the average values of pulse duration and pulse spacing for random pulse waveforms (12:44). Random pulse waveforms

may be used alone as a jamming source. A series of random pulses can also be used to modulate other noise waveforms.

A continuous wave signal (single frequency tone) can also be a source of interference to a radar receiver. For any given frequency, the CW tone can have a randomly varying amplitude and phase. Jordan and Penney present an expression for the PDF of the voltage of the CW interference (9:46-47). The PDF of the amplitude of the CW interference follows a somewhat parabolic distribution between the peak voltage values (9:46-47). For the density of the phase of CW interference, in the case of a pulsed radar it is equally likely that the CW interference is in phase with the received radar pulse as it is out of phase with the pulse (11.336). As is the case for random pulses, CW tones can be used alone as a jamming signal.

## 2.2 Jamming Waveforms

There are several types of waveforms employed in noise jammers. This section will discuss three main types of barrage jamming waveforms. Barrage jamming refers to radar jamming over a broad band of frequencies, as opposed to spot jamming which is defined as the jamming of a specific frequency (6:253,268). These barrage jamming waveforms are direct noise amplified (DINA), frequency modulated (FM) by-noise and amplitude modulated (AM) by-noise. The majority of what follows in this section is taken from a chapter of *Electronic Countermeasures* entitled "Effectiveness of Jamming Signals" by Bennighof and others (1:14-1-14-65).

DINA is merely amplified, bandlimited Gaussian noise (1:14-9). Over the passband of interest, DINA can be considered white (uniform PSD) (1:14-9). DINA can be easily generated by amplifying low level thermal noise (1:14-9-14-10). DINA has all of the properties of white Gaussian noise. For the purpose of analyzing the effects of DINA on radar receivers, the noise can be considered to be additive. That is, the noise is added to the radar signal prior to reception. The analysis of the effects of additive white Gaussian noise (AWGN) is presented in several radar texts. Skolnik

gives an analysis of the effects of AWGN on a radar's probability of detection and probability of false alarm (18:23-29). There are other jamming waveforms employed in radar jammers that are not truly Gaussian, but approach a true white Gaussian noise source.

In addition to DINA, FM-by-noise is one frequently employed class of noise jamming waveforms. There are two distinct types of FM-by-noise: FM-by-WB (wideband) noise, and FM-by-LF (low-frequency) noise. (1:14-10). Each type of FM-by-noise waveform has different effects on radar receivers. FM-by-noise is used extensively in current radar jammers.

One frequently used noise jamming waveform is FM-by-WB noise. FM-by-WB noise produces a spectrum nearly the same as that of DINA. (1:14-10). Bennighof and others showed that the noise output of a radar receiver's IF filter due to FM-by-WB noise can be the same as the IF filter output due to DINA (1:14-10). The effects of FM-by-WB noise are the same as those of DINA when the bandwidth of the barrage jamming waveform is greater than the noise bandwidth, and greater than the receiver bandwidth (1:14-22). Another FM-by-noise waveform used in several noise jammers is FM-by-LF noise.

FM-by-LF noise is produced in much the same way as FM-by-WB noise, but the noise signal is confined to a much narrower modulating bandwidth, a bandwidth less than the victim radar's bandwidth (1:14-10). Bennighof and others point out that FM-by-LF noise has two main advantages: FM-by-LF noise produces more power out of the victim radar's IF filter than does FM-by-WB noise, and FM-by-LF noise is more effective in the jamming of radar systems which use plan position indicator (PPI) displays (1:14-10). They also point out that FM-by-LF noise has one main disadvantage, that its effects are easily negated by electronic counter-countermeasures (ECCM) (1:14-10).

FM-by-noise is an extremely popular jamming waveform. Cassara and others describe a technique to generate a uniform PSD jamming signal from a Gaussian source using an FM-by-noise method. (2:330). The technique they present is to pass Gaussian noise through a nonlinear network,

followed by a frequency modulator (2:330-332). The result of their work was an extremely good source of uniform PSD noise over an electronically controlled bandwidth (2:332). The analysis presented by Knorr and Karantanas was mainly geared towards the study and optimization of FM-by-noise waveforms (10:273-277). The results of their study gave an indication of methods to improve the noise quality of an FM-by-noise waveform (10:275-277).

Another jamming waveform, although used much less frequently than FM-by-noise, is AM-by-noise. The AM-by-noise effects are similar to those of DINA, but it is difficult to produce wideband noise in this manner (1:14-11). This difficulty arises because the bandwidth of the AM-by-noise waveform is limited to twice the bandwidth of the modulating noise (1:14-11). This bandwidth limitation results from the property that the bandwidth of any AM (double sideband) signal is equal to twice the bandwidth of the modulating signal. Another reason that AM-by-noise is not as popular for use in jammers is that part of the total power transmitted by an AM-by-noise jammer is lost in the carrier (assuming large-carrier AM). In fact, no more than 50 percent of the total power will be in the jamming portion of the signal. The remaining power is in the carrier (12:37).

A further jamming technique is one that employs random pulse modulation. With this method, either an FM-by-WB noise or DINA waveform is pulse modulated. (1:14-11). The resulting random pulse jamming waveform employs random duration, spacing, and amplitude of the pulses, where the average pulse duration should be the same as the victim radar's pulse width (1:14-11). The effects of the random pulse jamming waveform are similar to those produced by the FM-by-LF noise waveform (1:14-11). The most effective random pulse jamming signal is one in which the amplitude, pulse duration and pulse interval are all varied randomly. However, such a signal is difficult to generate (12:42). It is much simpler in practice to generate a signal with constant amplitude pulses where the duration and pulse interval are varied randomly (12:42).

### *2.3 Summary*

As this chapter has pointed out, a great deal of work has been accomplished in the area of noise jamming. This literature search has concentrated on the sources of electrical noise and on noise jamming waveforms. Equations for the PDF's and PFD's of several different noise sources have been developed. There has also been much effort in the area of analyzing noise jamming waveforms. The goal of this review was to determine the most current information on these two subjects.

### III. Power Spectra

One of the main parameters to consider when studying the effects of jamming on radar receivers is the power spectral density (PSD) of the noise waveform. The PSD describes the frequency distribution of power in the noise jamming waveform. In this chapter, the power spectra for three cases is developed. First, the PSD for the shot noise in a parallel-plate diode is derived. Next, the PSD for a series of pulses of random amplitudes and spacing is developed. From this PSD, the special cases of the PSD for random binary transmission and the boxcar spectrum are derived. Lastly, the power spectrum for the general case of a series of pulses with random amplitudes, spacing, and phases is derived. For this general case, the special cases of the PSD of the random binary transmission and of the random telegraph signal are developed.

#### 3.1 Shot Noise

In this section, the power spectral density of the shot noise in a temperature-limited parallel-plate thermionic vacuum tube diode is derived. Much of the development here follows the treatment of this subject by Davenport and Root (4:112-124).

The anode current pulse due to the travel of a single electron is represented by the following equation:

$$i_e(t) = \begin{cases} \frac{2q}{\tau_a} \left( \frac{t}{\tau_a} \right) & , \quad 0 \leq t \leq \tau_a \\ 0 & , \quad \text{elsewhere} \end{cases} \quad (3.1)$$

where

$q$  = charge per electron (  $-1.6 \times 10^{-19}$  Coulomb)

$\tau_a$  = transit time



The total current flowing through the vacuum tube diode is the sum of the individual electron current pulses. The total current in a time interval  $(-T, T)$  resulting from the flow of  $K$  electrons is represented by the following equation:

$$I(t) = \sum_{k=1}^K i_e (t - t_k) \quad \text{for } -T \leq t \leq T \quad (3.2)$$

where

$t_k$  = emission time of the  $k$ th electron

The number of electrons emitted in a given time interval is assumed to have a Poisson distribution. The probability of  $K$  electrons emitted in a time interval  $\tau$  is given by the following expression for the Poisson distribution:

$$P(K, \tau) = \frac{(\bar{n}\tau)^K \exp(-\bar{n}\tau)}{K!} \quad (3.3)$$

where

$\bar{n}$  = average number of electrons

The following statistical properties are obtained from the Poisson distribution:

$$E_{\tau}(K) = \bar{n}\tau \quad (3.4)$$

$$E_{\tau}(K^2) = \bar{n}\tau + (\bar{n}\tau)^2 \quad (3.5)$$

$$\text{Var}_{\tau}(K) = \bar{n}\tau \quad (3.6)$$

**3.1.1 Average Current** The total diode current  $I(t)$  is a function of the total number of electrons emitted,  $K$ , and the emission times of each electron, the  $t_k$ 's. These terms are all random variables. The average current is obtained from the expected value of the total current  $I(t)$ :

$$E[I(t)] = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} I(t) p(t_1, \dots, t_K, K) dt_1 \dots dt_K dK \quad (3.7)$$

The joint PDF of the number of electrons and the emission times may be written

$$p(t_1, \dots, t_K, K) = p(t_1, \dots, t_K | K) p(K) \quad (3.8)$$

The emission times, the  $t_k$ 's, are independent, uniformly distributed random variables with probability density function (4:117-119)

$$p(t_k) = \begin{cases} \frac{1}{2T} & , \quad t \leq t_k \leq t + 2T \\ 0 & , \quad \text{elsewhere} \end{cases} \quad (3.9)$$

Since the  $t_k$ 's are statistically independent random variables, the joint PDF of the emission times and the number of electrons becomes

$$p(t_1, \dots, t_K, K) = p(t_1, | K) \dots p(t_K, | K) p(K) \quad (3.10)$$

and the expected value of the total current can be written

$$\begin{aligned} E[I(t)] &= \int_{-\infty}^{\infty} p(K) dK \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} I(t) \frac{dt_1}{2T} \cdots \frac{dt_K}{2T} \\ &= \int_{-\infty}^{\infty} p(K) dK \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \sum_{k=1}^K i_e (t - t_k) \frac{dt_1}{2T} \cdots \frac{dt_K}{2T} \end{aligned}$$

$$\begin{aligned}
E[I(t)] &= \int_{-\infty}^{\infty} p(K) dK \left[ \sum_{k=1}^K \left( \frac{1}{2T} \right)^K \int_{-T}^T dt_1 \cdots \int_{-T}^T dt_{K-1} \int_{-T}^T i_e(t - t_k) dt_K \right] \\
&= \int_{-\infty}^{\infty} p(K) dK \left[ \sum_{k=1}^K \left( \frac{1}{2T} \right)^K (2T)^{K-1} q \right] \quad (3.11)
\end{aligned}$$

Let  $\bar{I}(t) = E[I(t)]$ . The expected value of the current (the DC current) is

$$\begin{aligned}
\bar{I} &= \int_{-\infty}^{\infty} p(K) dK \sum_{k=1}^K \frac{q}{2T} \\
&= \frac{q}{2T} \int_{-\infty}^{\infty} K p(K) dK \\
&= \frac{q}{2T} E_{2T}(K) \\
&= \bar{n}q \quad (3.12)
\end{aligned}$$

**3.1.2 Power Spectral Density** By the application of the Wiener-Khinchine relationship, which states that the power spectral density of a stationary process is obtained from the Fourier transform of the autocorrelation function, the power spectral density is obtained (17:145). The autocorrelation function is derived as follows:

$$\begin{aligned}
R_I(\tau) &= E[I(t)I(t + \tau)] \\
&= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} I(t)I(t + \tau) p(t_1, \dots, t_K, K) dt_1 \dots dt_K dK \\
&= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \sum_{i=1}^K i_e(t - t_i) \sum_{j=1}^K i_e(t + \tau - t_j) p(t_1, \dots, t_K, K) dt_1 \dots dt_K dK \quad (3.13)
\end{aligned}$$

Because the  $t_k$ 's are independent and uniformly distributed with the probability density function of Eq ( 3.9) the equation for the autocorrelation function may be written

$$\begin{aligned}
R_I(\tau) &= \int_{-\infty}^{\infty} p(K) dK \int_{-T}^T \cdots \int_{-T}^T \sum_{i=1}^K i_e(t-t_i) \sum_{j=1}^K i_e(t+\tau-t_j) \frac{dt_1}{2T} \cdots \frac{dt_K}{2T} \\
&= \int_{-\infty}^{\infty} p(K) dK \left[ \sum_{i=1}^K \sum_{j=1}^K \int_{-T}^T \frac{dt_1}{2T} \cdots \int_{-T}^T i_e(t-t_i) i_e(t+\tau-t_j) \frac{dt_K}{2T} \right] \quad (3.14)
\end{aligned}$$

There are  $K^2$  terms under the double summation:  $K$  terms when  $i = j$ , and  $K^2 - K$  terms when  $i \neq j$ .

For  $i = j$

$$\begin{aligned}
\int_{-T}^T \frac{dt_1}{2T} \cdots \int_{-T}^T i_e(t-t_i) i_e(t+\tau-t_j) \frac{dt_K}{2T} &= \left( \frac{1}{2T} \right)^K (2T)^{K-1} \int_{-T}^T i_e(t) i_e(t+\tau) dt \\
&= \frac{1}{2T} \int_{-T}^T i_e(t) i_e(t+\tau) dt \quad (3.15)
\end{aligned}$$

For  $i \neq j$

$$\begin{aligned}
\int_{-T}^T \frac{dt_1}{2T} \cdots \int_{-T}^T i_e(t-t_i) i_e(t+\tau-t_j) \frac{dt_K}{2T} &= \left( \frac{1}{2T} \right)^K (2T)^{K-2} \int_{-T}^T i_e(t-t_i) dt_i \\
&\quad \times \int_{-T}^T i_e(t+\tau-t_j) dt_j \\
&= \left( \frac{1}{2T} \right)^2 \int_{-T}^T i_e(t) dt \int_{-T}^T i_e(t) dt \\
&= q^2 \left( \frac{1}{2T} \right)^2 \quad (3.16)
\end{aligned}$$

Combining the above two cases, the autocorrelation function of the total current is now

$$\begin{aligned}
R_I(\tau) &= \int_{-\infty}^{\infty} \left[ (K^2 - K) \left( \frac{q}{2T} \right)^2 + \frac{K}{2T} \int_{-\infty}^{\infty} i_e(t) i_e(t + \tau) dt \right] p(K) dK \\
&= \left( \frac{q}{2T} \right)^2 \int_{-\infty}^{\infty} (K^2 - K) p(K) dK \\
&\quad + \left[ \frac{1}{2T} \int_{-\infty}^{\infty} i_e(t) i_e(t + \tau) dt \right] \int_{-\infty}^{\infty} K p(K) dK \\
&= \left( \frac{q}{2T} \right)^2 E_{2T} (K^2 - K) + \frac{1}{2T} E_{2T} (K) \int_{-\infty}^{\infty} i_e(t) i_e(t + \tau) dt \quad (3.17)
\end{aligned}$$

Using Eqs ( 3.4) and ( 3.5) the expectations are

$$E_{2T}(K) = 2T\bar{n} \quad (3.18)$$

$$E_{2T} (K^2 - K) = (2T\bar{n})^2 \quad (3.19)$$

With these expectations, the autocorrelation function is

$$\begin{aligned}
R_I(\tau) &= \left( \frac{q}{2T} \right)^2 (2T\bar{n})^2 + \frac{1}{2T} (2T\bar{n}) \int_{-\infty}^{\infty} i_e(t) i_e(t + \tau) dt \\
&= (q\bar{n})^2 + \bar{n} \int_{-\infty}^{\infty} i_e(t) i_e(t + \tau) dt \quad (3.20)
\end{aligned}$$

The shot noise current of interest is the AC portion of the total current (the fluctuation about the mean value). Let  $i(t)$  be the AC diode current:

$$i(t) = I(t) - \bar{I}(t) \quad (3.21)$$

The autocorrelation function of the AC current is

$$R_i(\tau) = \bar{n} \int_{-\infty}^{\infty} i_e(t) i_e(t + \tau) dt \quad (3.22)$$

Since the shot noise is a stationary process, the Fourier transform of the autocorrelation function of the process yields the power spectral density of the shot noise:

$$\begin{aligned} S_i(f) &= \int_{-\infty}^{\infty} R_i(\tau) e^{-j2\pi f\tau} d\tau \\ &= \int_{-\infty}^{\infty} \bar{n} \int_{-\infty}^{\infty} i_e(t) i_e(t + \tau) dt e^{-j2\pi f\tau} d\tau \end{aligned} \quad (3.23)$$

Let  $\tau = t' - t$ , so the power spectral density now becomes

$$\begin{aligned} S_i(f) &= \bar{n} \int_{-\infty}^{\infty} i_e(t) e^{-j2\pi f(t' - t)} dt \int_{-\infty}^{\infty} i_e(t') dt' \\ &= \bar{n} \int_{-\infty}^{\infty} i_e(t) e^{j2\pi ft} dt \int_{-\infty}^{\infty} i_e(t') e^{-j2\pi ft'} dt' \end{aligned} \quad (3.24)$$

Let  $G(f)$  be the Fourier transform of the current pulse:

$$G(f) = \int_{-\infty}^{\infty} i_e(t) e^{-j\omega t} dt \quad (3.25)$$

where

$$\omega = 2\pi f$$

so that the power spectral density is

$$S_i(\omega) = \bar{n} |G(\omega)|^2 \quad (3.26)$$

For low frequency values ( $f \ll 1/\tau_a$ ) the PSD is approximately

$$S_i(\omega) \approx \bar{n} |G(0)|^2 \quad (3.27)$$

From Eqs ( 3.1) and ( 3.25)  $G(0) = q$  and it follows from Eq ( 3.12) that the power spectral density is approximated by

$$S_i(\omega) \approx \bar{I}q \quad (3.28)$$

which is the Schottky formula.

**3.1.3 The Power Spectral Density for the Parallel-Plane Diode** In this section the power spectral density for the shot noise from the parallel-plane diode will be developed. The expression for the power spectral density is given by Eq ( 3.26). The current pulse is defined by Eq ( 3.1) and the Fourier transform of the current pulse is

$$\begin{aligned} G(\omega) &= \int_0^{\tau_a} \frac{2q}{\tau_a^2} t e^{-j\omega t} dt \\ &= \frac{2q}{(\omega\tau_a)^2} (1 - e^{-j\omega\tau_a} - j\omega\tau_a e^{-j\omega\tau_a}) \\ &= \frac{2q}{(\omega\tau_a)^2} [(1 - \cos\omega\tau_a - \omega\tau_a \sin\omega\tau_a) + j(\sin\omega\tau_a - \omega\tau_a \cos\omega\tau_a)] \end{aligned} \quad (3.29)$$

The square of the magnitude of the Fourier transform of the current pulse is

$$\begin{aligned} |G(\omega)|^2 &= \frac{4q^2}{(\omega\tau_a)^4} [(1 - \cos\omega\tau_a - \omega\tau_a \sin\omega\tau_a)^2 + (\sin\omega\tau_a - \omega\tau_a \cos\omega\tau_a)^2] \\ &= \frac{4q^2}{(\omega\tau_a)^4} [(\omega\tau_a)^2 + 2(1 - \cos\omega\tau_a - \omega\tau_a \sin\omega\tau_a)] \end{aligned} \quad (3.30)$$

Substituting the above equation into Eq ( 3.26) the power spectral density becomes

$$S_i(\omega) = \frac{4q^2\bar{n}}{(\omega\tau_a)^4} \left[ (\omega\tau_a)^2 + 2(1 - \cos\omega\tau_a - \omega\tau_a \sin\omega\tau_a) \right] \quad (3.31)$$

From Eq ( 3.12)  $\bar{n}q = \bar{I}$  and the power spectral density becomes

$$S_i(\omega) = \frac{4q\bar{I}}{(\omega\tau_a)^4} \left[ (\omega\tau_a)^2 + 2(1 - \cos\omega\tau_a - \omega\tau_a \sin\omega\tau_a) \right] \quad (3.32)$$

which is illustrated in Figure 3.1.

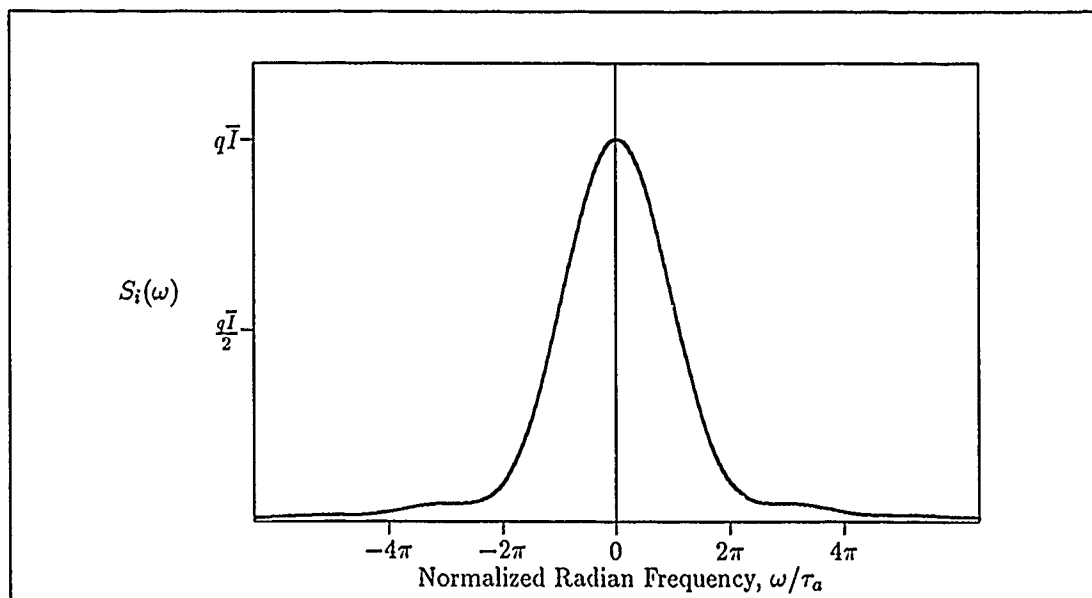


Figure 3.1. Parallel-Plane Diode Shot Noise PSD

This equation agrees with the expressions for the PSD derived by Davenport and Root (4:124) and Papoulis (14:357-360).



### 3.2 Pulses with Random Amplitudes and Intervals

The power spectral density for a series of pulses with random amplitudes and pulse intervals is developed here. An infinite series of pulses with random amplitudes and intervals is defined by the following equation:

$$y(t) = \sum_{k=-\infty}^{\infty} a_k g(t - kT_o - \epsilon_k) \quad (3.33)$$

where  $a_k$  is the pulse amplitude with PDF  $p(a)$ ,  $T_o$  is the average pulse repetition interval, and  $\epsilon_k$  is the deviation from  $T_o$  of the  $k$ th pulse with PDF  $p(\epsilon)$  and  $E(\epsilon) = 0$ . The series is truncated to a finite interval from  $-NT_o$  to  $+NT_o$ :

$$y_N(t) = \sum_{k=-N}^N a_k g(t - kT_o - \epsilon_k) \quad (3.34)$$

A portion of a series of pulses with random amplitudes and spacing is illustrated in Figure 3.2.

The Fourier transform of the sequence is

$$Y_N(f) = G(f) \sum_{k=-N}^N a_k e^{-j2\pi f(kT_o + \epsilon_k)} \quad (3.35)$$

where the Fourier transform of a single pulse is

$$G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi f t} dt \quad (3.36)$$

**3.2.1 Power Spectral Density** The expected value of the power spectral density is derived here. The expected value will be used since the  $a_k$  and  $\epsilon_k$  terms are random variables. The average PSD is defined by

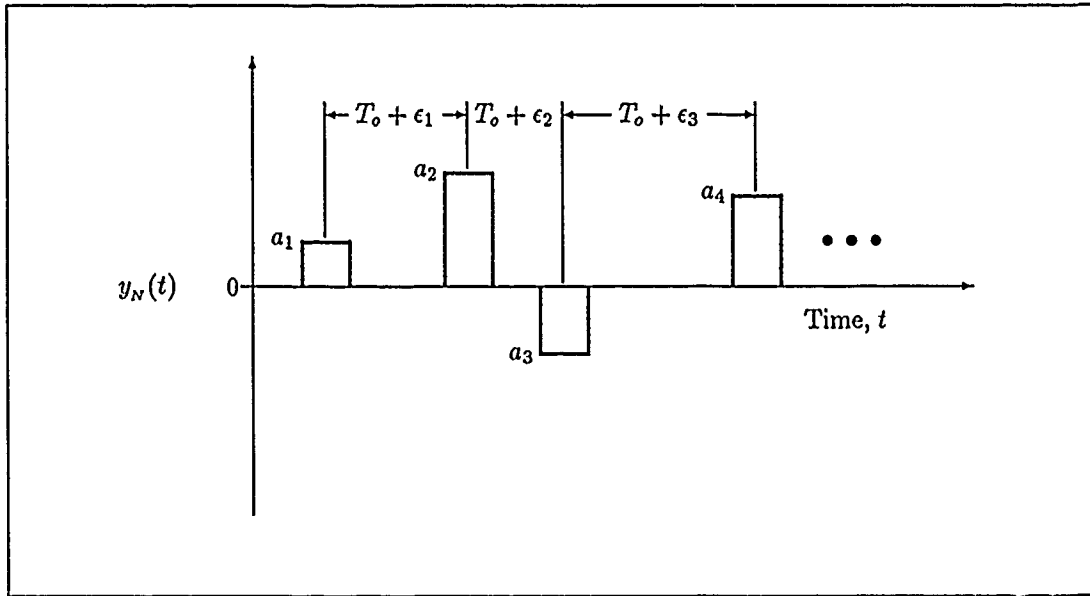


Figure 3.2. Series of Pulses with Random Amplitudes and Spacing

$$\overline{S_Y(f)} = \lim_{N \rightarrow \infty} \frac{1}{(2N+1)T_0} \overline{|Y_N(f)|^2} \quad (3.37)$$

where there are approximately  $2N+1$  pulses between  $-NT_0$  and  $+NT_0$ . The square of the magnitude of the Fourier transform of the series of pulses is represented by the following equation:

$$|Y_N(f)|^2 = |G(f)|^2 \sum_{k=-N}^N \sum_{l=-N}^N a_k a_l e^{-j2\pi f(kT_0 + \epsilon_k)} e^{+j2\pi f(lT_0 + \epsilon_l)} \quad (3.38)$$

The expected value of the above equation is

$$\begin{aligned} \overline{|Y_N(f)|^2} &= |G(f)|^2 \sum_{k=-N}^N \sum_{l=-N}^N [e^{-j2\pi f k T_0} e^{+j2\pi f l T_0} \\ &\quad \times E(a_k a_l e^{-j2\pi f \epsilon_k} e^{+j2\pi f \epsilon_l})] \end{aligned} \quad (3.39)$$

Since the  $a_k$ 's and  $\epsilon_k$ 's are statistically independent the expectation in the above expression has the following values

$$E(a_k a_l e^{-j2\pi f \epsilon_k} e^{j2\pi f \epsilon_l}) = \begin{cases} E(a^2) & , \quad k = l \\ E^2(a) |\Phi_\epsilon(f)|^2 & , \quad k \neq l \end{cases} \quad (3.40)$$

where

$$\Phi_\epsilon(f) = \int_{-\infty}^{\infty} p(\epsilon) e^{j2\pi f \epsilon} d\epsilon \quad (3.41)$$

which is the characteristic function of the deviation,  $\epsilon$ .

There are  $2N + 1$  terms where  $k = l$ . Therefore the expression for the square of the magnitude of the Fourier transform becomes

$$\overline{|Y_N(f)|^2} = |G(f)|^2 \left[ (2N + 1)E(a^2) + E^2(a) |\Phi_\epsilon(f)|^2 \sum_{k \neq l} \sum e^{-j2\pi f k T_0} e^{j2\pi f l T_0} \right] \quad (3.42)$$

The double summation in the above equation may be written

$$\begin{aligned} \sum_{k \neq l} \sum e^{-j2\pi f k T_0} e^{j2\pi f l T_0} &= -(2N + 1) + \sum_{k=-N}^N \sum_{l=-N}^N e^{-j2\pi f k T_0} e^{j2\pi f l T_0} \\ &= -(2N + 1) + \left| \sum_{k=-N}^N e^{-j2\pi f k T_0} \right|^2 \end{aligned} \quad (3.43)$$

so that the expected value of the magnitude of the square of the Fourier transform now becomes

$$\begin{aligned} \overline{|Y_N(f)|^2} &= |G(f)|^2 \left\{ (2N+1) \left[ E(a^2) - E^2(a) |\Phi_\epsilon(f)|^2 \right] \right. \\ &\quad \left. + E^2(a) |\Phi_\epsilon(f)|^2 \left| \sum_{k=-N}^N e^{-j2\pi f k T_o} \right|^2 \right\} \end{aligned} \quad (3.44)$$

The average power spectral density is arrived at by taking the limit of the above expression as  $N$  approaches infinity:

$$\begin{aligned} \overline{S_Y(f)} &= \lim_{N \rightarrow \infty} \frac{1}{(2N+1)T_o} |G(f)|^2 \left\{ (2N+1) \left[ E(a^2) - E^2(a) |\Phi_\epsilon(f)|^2 \right] \right. \\ &\quad \left. + E^2(a) |\Phi_\epsilon(f)|^2 \left| \sum_{k=-N}^N e^{-j2\pi f k T_o} \right|^2 \right\} \\ &= \frac{1}{T_o} |G(f)|^2 \left\{ E(a^2) - E^2(a) |\Phi_\epsilon(f)|^2 \right. \\ &\quad \left. + E^2(a) |\Phi_\epsilon(f)|^2 \lim_{N \rightarrow \infty} \frac{1}{2N+1} \left| \sum_{k=-N}^N e^{-j2\pi f k T_o} \right|^2 \right\} \end{aligned} \quad (3.45)$$

The summation in the above equation may be expanded using Euler's formula and can be written

$$\begin{aligned} \sum_{k=-N}^N e^{-j2\pi f k T_o} &= 1 + \sum_{k=1}^N e^{-j2\pi f k T_o} + \sum_{k=-N}^{-1} e^{-j2\pi f k T_o} \\ &= 1 + \sum_{k=1}^N e^{-j2\pi f k T_o} + \sum_{k=1}^N e^{+j2\pi f k T_o} \\ &= 1 + \sum_{k=1}^N (e^{-j2\pi f k T_o} + e^{+j2\pi f k T_o}) \\ &= 1 + 2 \sum_{k=1}^N \cos 2\pi f k T_o \end{aligned} \quad (3.46)$$

The following expression for the sum of cosines is given by Eq (1.342,2) from Gradshteyn and Ryzhik (8:30):

$$\sum_{k=0}^n \cos kx = \cos \frac{n+1}{2}x \sin \frac{nx}{2} \operatorname{cosec} \frac{x}{2} + 1 \quad (3.47)$$

which can be manipulated to the form

$$\sum_{k=1}^N \cos kx = \frac{\cos \frac{N+1}{2}x \sin \frac{Nx}{2}}{\sin \frac{x}{2}} \quad (3.48)$$

Using Eq (3.48) the summation in Eq (3.46) becomes

$$\sum_{k=-N}^N e^{-j2\pi f k T_o} = 1 + \frac{2 \cos \left( \frac{N+1}{2} \right) 2\pi f k T_o \sin \frac{N2\pi f k T_o}{2}}{\sin \frac{2\pi f k T_o}{2}} \quad (3.49)$$

Using the trigonometric identity

$$\cos A \sin B = \frac{1}{2} \sin (A + B) - \frac{1}{2} \sin (A - B) \quad (3.50)$$

the summation can be expressed as

$$\begin{aligned} \sum_{k=-N}^N e^{-j2\pi f k T_o} &= 1 + \frac{\sin (2N+1)\pi f T_o - \sin \pi f T_o}{\sin \pi f T_o} \\ &= \frac{\sin \pi f T_o + \sin (2N+1)\pi f T_o - \sin \pi f T_o}{\sin \pi f T_o} \\ &= \frac{\sin (2N+1)\pi f T_o}{\sin \pi f T_o} \end{aligned} \quad (3.51)$$

The limit now becomes

$$\lim_{N \rightarrow \infty} \frac{1}{2N+1} \left| \sum_{k=-N}^N e^{-j2\pi f k T_o} \right|^2 = \lim_{N \rightarrow \infty} \frac{1}{2N+1} \left[ \frac{\sin(2N+1)\pi f T_o}{\sin \pi f T_o} \right]^2 \quad (3.52)$$

By letting

$$\Omega = (2N+1)$$

$$\alpha = \pi f T_o$$

the limit can be expressed

$$\lim_{N \rightarrow \infty} \frac{1}{2N+1} \left| \sum_{k=-N}^N e^{-j2\pi f k T_o} \right|^2 = \lim_{\Omega \rightarrow \infty} \frac{1}{\Omega} \left[ \frac{\Omega}{\pi} \left( \frac{\sin \alpha \Omega}{\alpha \Omega} \right)^2 \right] \left( \frac{\pi \alpha^2 \Omega}{\sin^2 \alpha} \right) \quad (3.53)$$

Using the following expression for the limit of the square of the sinc-over-argument in the above equation

$$\lim_{\Omega \rightarrow \infty} \frac{\Omega}{\pi} \left( \frac{\sin \alpha \Omega}{\alpha \Omega} \right)^2 = \delta(\alpha) \quad (3.54)$$

leads to the following form

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{1}{2N+1} \left| \sum_{k=-N}^N e^{-j2\pi f k T_o} \right|^2 &= \frac{\pi \alpha^2}{\sin^2 \alpha} \delta(\alpha) \\ &= \pi \delta(\alpha) \frac{\alpha^2}{\sin^2 \alpha} \Big|_{\alpha=0} \end{aligned} \quad (3.55)$$

By L'Hôpital's rule, the following result is achieved:

$$\lim_{\alpha \rightarrow 0} \frac{\alpha^2}{\sin^2 \alpha} = 1 \quad (3.56)$$

which leads to the following equation:

$$\lim_{N \rightarrow \infty} \frac{1}{2N+1} \left| \sum_{k=-N}^N e^{-j2\pi f k T_o} \right|^2 = \pi \delta(\pi f T_o) \quad (3.57)$$

Using the following property of the delta function

$$\delta(\alpha f) = \frac{1}{|\alpha|} \delta(f) \quad (3.58)$$

the limit can now be written

$$\lim_{N \rightarrow \infty} \frac{1}{2N+1} \left| \sum_{k=-N}^N e^{-j2\pi f k T_o} \right|^2 = \frac{1}{T_o} \delta(f) \quad (3.59)$$

However, the expression

$$\frac{\sin(2N+1)\pi f T_o}{\sin \pi f T_o} \quad (3.60)$$

is periodic, with period  $T_o$ . So, the limit of the above summation becomes

$$\lim_{N \rightarrow \infty} \frac{1}{2N+1} \left| \sum_{k=-N}^N e^{-j2\pi f k T_o} \right|^2 = \frac{1}{T_o} \sum_{n=0}^{\infty} \delta\left(f - \frac{n}{T_o}\right) \quad (3.61)$$

Substituting the above expression into Eq ( 3.45) the average power spectral density becomes

$$\overline{S_Y(f)} = \frac{1}{T_o} |G(f)|^2 \left\{ E(a^2) - E^2(a) |\Phi_e(f)|^2 + \frac{1}{T_o} E^2(a) |\Phi_e(f)|^2 \sum_{n=0}^{\infty} \delta\left(f - \frac{n}{T_o}\right) \right\} \quad (3.62)$$

This expression for the power spectral density is similar to those derived by Lawson and Uhlenbeck (11:43-44). However, this expression takes into account both random amplitudes and random intervals, where Lawson and Uhlenbeck have separate equations for each case. Also, the

magnitude of the expression derived here is one half the magnitude of the expressions derived by Lawson and Uhlenbeck. The difference is because Lawson and Uhlenbeck use a one-sided spectrum, whereas the spectrum derived here is two-sided.

**3.2.2 Special Cases** In this section, two special cases of PSD of a series of pulses with random amplitudes and spacing. First, the power spectral density of the random binary transmission is derived. This derivation is followed by the development of the boxcar spectrum.

**3.2.2.1 Random Binary Transmission** One special case of the expression for the power spectral density of pulses with random amplitudes and spacing is the random binary transmission as defined by Papoulis (14:294, 341). The amplitudes are equally likely, and distributed according to

$$P(a = 1) = P(a = -1) = \frac{1}{2} \quad (3.63)$$

Which leads to the following expected values:

$$\begin{aligned} E(a) &= 0 \\ E(a^2) &= 1 \end{aligned} \quad (3.64)$$

The Fourier transform of a single square pulse is given by

$$G(f) = T \left[ \frac{\sin(\pi f T)}{\pi f T} \right] \quad (3.65)$$

where  $T$  is the pulse duration. In this case  $T$  is a fixed value. Therefore, the deviation,  $\epsilon$  is zero. This implies that  $p(\epsilon) = \delta(\epsilon)$ , and  $\Phi_\epsilon(f)$  is unity. From Eq ( 3.62) this leads to



$$S(f) = \frac{1}{T} |G(f)|^2 \left[ E(a^2) - E^2(a) + \frac{1}{T} E^2(a) \sum_{n=0}^{\infty} \delta \left( f - \frac{n}{T} \right) \right] \quad (3.66)$$

Substituting Eqs ( 3.64) and ( 3.65) into the above equation yields

$$\begin{aligned} S(f) &= \frac{1}{T} \left[ \frac{\sin(\pi f T)}{\pi f} \right]^2 \\ &= \frac{\sin^2(\pi f T)}{\pi^2 f^2 T} \end{aligned} \quad (3.67)$$

This equation for the power spectral density is illustrated in Figure 3.3. This expression agrees with the power spectral density for the random binary transmission presented by Papoulis (14:341).

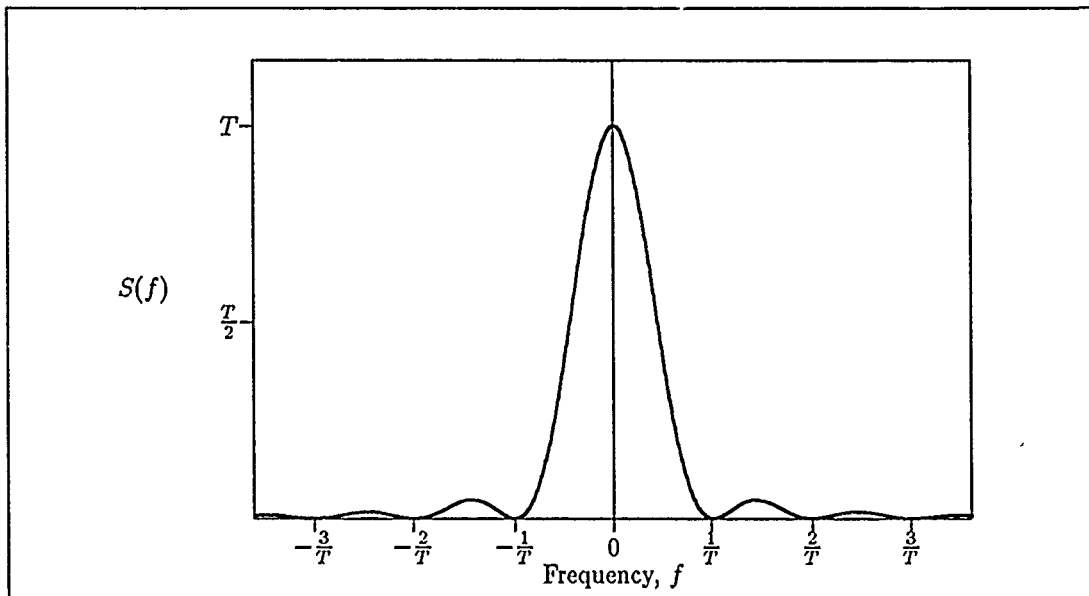


Figure 3.3. Random Binary Transmission PSD

**3.2.2.2 Boxcar Spectrum** The special case of the boxcar spectrum is studied here. The spectrum considered here is the spectrum of the output of a boxcar generator (more commonly

referred to as a sample-and-hold circuit). The output of the boxcar generator is a series of square pulses with random amplitudes, distributed according to  $p(a)$ , and with constant width  $T$ . The Fourier transform of a single, unit amplitude square pulse of width  $T$  is

$$G(f) = T \left( \frac{\sin \pi f T}{\pi f T} \right) \quad (3.68)$$

As was the case for the random binary transmission, the pulse width is a fixed value, and the deviation,  $\epsilon$ , is distributed according to

$$p(\epsilon) = \delta(\epsilon) \quad (3.69)$$

which implies that  $\Phi_\epsilon(f) = 1$ . Substituting this value for  $\Phi_\epsilon(f)$  and Eq (3.68) into Eq (3.62) the power spectral density becomes

$$S(f) = T \left( \frac{\sin \pi f T}{\pi f T} \right)^2 \left\{ E(a^2) - E^2(a) + \frac{1}{T} E^2(a) \sum_{n=0}^{\infty} \delta \left( f - \frac{n}{T} \right) \right\} \quad (3.70)$$

The following relationship exists for various values of  $\frac{n}{T}$ :

$$\left. \frac{\sin \pi f T}{\pi f T} \right|_{f=\frac{n}{T}} = \begin{cases} 1 & , \quad n = 0 \\ 0 & , \quad n = 1, 2, 3, \dots \end{cases} \quad (3.71)$$

The above relationship implies that the delta function in the equation for the PSD only applies when  $f = 0$ . Therefore the PSD is written

$$S(f) = \frac{\sin^2 \pi f T}{(\pi f)^2 T} \left\{ [E(a^2) - E^2(a)] + E^2(a) \frac{1}{T} \delta(f) \right\} \quad (3.72)$$

where  $E(a^2) - E^2(a)$  is recognized as the variance of the amplitude  $a$ . This spectrum agrees with the power spectrum for this case as presented in Lawson and Uhlenbeck (11:274).

### 3.3 Pulses with Random Amplitudes, Durations, and Phases

In the previous section, the power spectral density of a series of pulses with random amplitudes and durations was developed. In this section, the power spectral density for the more general case of a series of pulses with random amplitudes, durations and phases is developed. The development here follows closely that of Lawson and Uhlenbeck (11:44-46). A single pulse is defined by the expression  $a_n e^{j2\pi f_0 t}$ . Further, the phase will change for each pulse, with  $p(\alpha)$  the probability density function of the phase,  $\alpha_k$ . Therefore, at the following times the pulse  $y(t)$  is defined by

$$\begin{aligned} t_1 &: y(t) = a_1 e^{j2\pi(f_0 t + \alpha_1)} \\ t_2 &: y(t) = a_2 e^{j2\pi(f_0 t + \alpha_1 + \alpha_2)} \\ &\vdots \\ t_n &: y(t) = a_n e^{j2\pi(f_0 t + \alpha_1 + \alpha_2 + \dots + \alpha_n)} \end{aligned} \quad (3.73)$$

where the  $a_k$ 's are real, statistically independent random variables distributed with PDF  $p(a)$ . The amplitudes have the following joint expectations:

$$E(a_k a_l) = \begin{cases} E(a^2) & , \quad k = l \\ E^2(a) & , \quad k \neq l \end{cases} \quad (3.74)$$

The length of each pulse is distributed according to the PDF  $p(l)$  and is defined by

$$l_k = \text{the spacing between times } t_k \text{ and } t_{k+1} \quad (3.75)$$

so that the time of occurrence of the  $k$ th pulse is

$$t_k = t_1 + l_1 + l_2 + \cdots + l_{k-1} \quad (3.76)$$

An example of a series of pulses with random amplitudes, phases and spacing is illustrated in Figure 3.4.

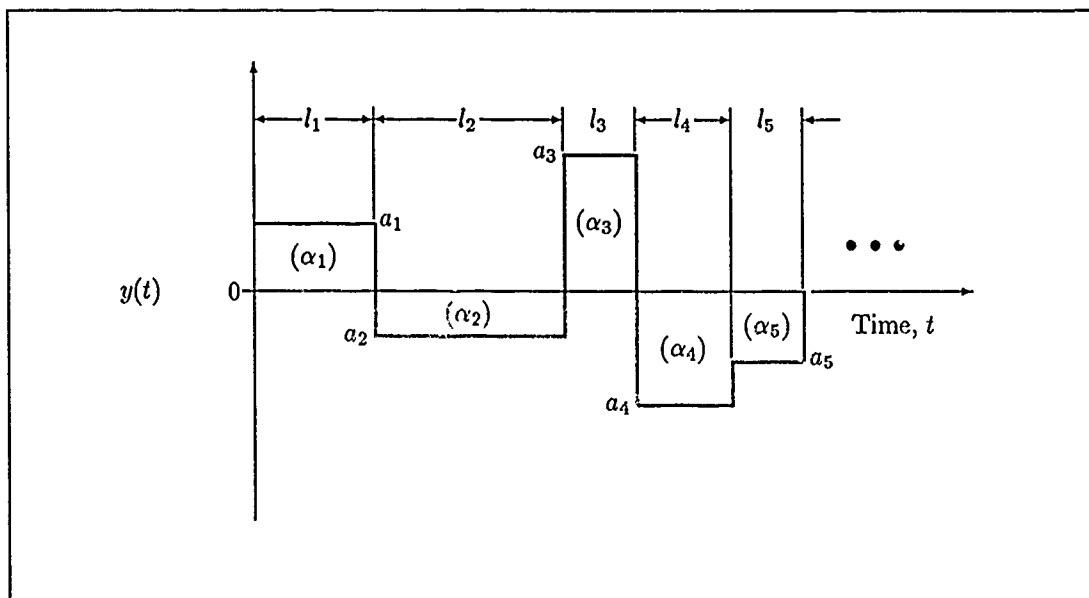


Figure 3.4. Series of Pulses with Random Amplitudes, Phases and Spacing

Some useful relationships presented by Lawson and Uhlenbeck are expanded here (11:45):

$$\Theta = \int_{-\frac{1}{2}}^{\frac{1}{2}} p(\alpha) e^{j2\pi\alpha} d\alpha = A + jB = E(e^{j\beta}) \quad (3.77)$$

$$\Omega = \int_0^{\infty} p(l) e^{-j2\pi l(f-f_0)} dl = \phi(f) + j\psi(f) = E(e^{-j\omega' l}) \quad (3.78)$$

$$\bar{l} = \int_0^{\infty} p(l) l dl = E(l) \quad (3.79)$$

where

$$\beta_n = 2\pi\alpha_n$$

$$\omega_c = 2\pi(f - f_o)$$

Some relationships involving  $\Omega$  are presented here. Using Eq ( 3.76)

$$\begin{aligned} E(e^{j\omega_c t_k}) &= E \left[ e^{j\omega_c(t_1 + l_1 + l_2 + \dots + l_{k-1})} \right] \\ &= e^{j\omega_c t_1} E \left[ e^{j\omega_c(l_1 + l_2 + \dots + l_{k-1})} \right] \\ &= e^{j\omega_c t_1} \Omega^{k-1} \end{aligned} \quad (3.80)$$

since the  $l_n$ 's are statistically independent. Further, the expectation of the conjugate of the previous equation is

$$\begin{aligned} E(e^{-j\omega_c t_k}) &= E \left[ (e^{j\omega_c t_k})^* \right] \\ &= e^{j\omega_c t_1} (\Omega^*)^{k-1} \end{aligned} \quad (3.81)$$

Finally, combining Eqs ( 3.80) and ( 3.81) the following relationship results:

$$E \left[ e^{j\omega_c(t_k - t_l)} \right] = \begin{cases} 1 & , \quad k = l \\ (\Omega^*)^{l-k} & , \quad k < l \\ \Omega^{k-l} & , \quad k > l \end{cases} \quad (3.82)$$

Likewise, some relationships involving  $\Theta$  are presented here:

$$E \left[ e^{j(\beta_1 + \beta_2 + \dots + \beta_k)} \right] = \Theta^k \quad (3.83)$$

since the  $\alpha_n$ 's and therefore the  $\beta_n$ 's are statistically independent. Now combining Eq ( 3.83) with its conjugate, the following relationship is arrived at:

$$E \left[ e^{j(\beta_1 + \beta_2 + \dots + \beta_k)} e^{-j(\beta_1 + \beta_2 + \dots + \beta_l)} \right] = \begin{cases} 1 & , \quad k = l \\ (\Theta^*)^{l-k} & , \quad k < l \\ \Theta^{k-l} & , \quad k > l \end{cases} \quad (3.84)$$

A single pulse from time  $t_k$  to time  $t_{k+1}$  is expressed as

$$y(t) = a_k e^{j2\pi(f_o t + \alpha_1 + \alpha_2 + \dots + \alpha_k)} \quad (3.85)$$

The Fourier transform of this single pulse from  $t_k$  to  $t_{k+1}$  is

$$\begin{aligned} Y(f; t_k, t_{k+1}) &= a_k e^{j(\beta_1 + \beta_2 + \dots + \beta_k)} \int_{t_k}^{t_{k+1}} e^{-j2\pi(f - f_o)t} dt \\ &= -\frac{1}{j\omega_c} a_k e^{j(\beta_1 + \beta_2 + \dots + \beta_k)} (e^{-j\omega_c t_{k+1}} - e^{-j\omega_c t_k}) \end{aligned} \quad (3.86)$$

A finite series of N pulses from time  $t_1$  to  $t_{N+1}$  is

$$y_N(t) = \sum_{k=1}^N a_k e^{j2\pi(f_o t + \alpha_1 + \alpha_2 + \dots + \alpha_k)} \quad (3.87)$$

and the Fourier transform of this series of pulses is

$$\begin{aligned}
Y_N(f) &= -\frac{1}{j\omega_c} \sum_{k=1}^N a_k e^{j(\beta_1+\beta_2+\dots+\beta_k)} (e^{-j\omega_c t_{k+1}} - e^{-j\omega_c t_k}) \\
&= -\frac{1}{j\omega_c} \sum_{k=1}^N A_k
\end{aligned} \tag{3.88}$$

where

$$A_k = a_k e^{j(\beta_1+\beta_2+\dots+\beta_k)} (e^{-j\omega_c t_{k+1}} - e^{-j\omega_c t_k}) \tag{3.89}$$

**3.3.1 Power Spectral Density** The average power spectral density of the series is developed here. The average PSD is defined by

$$\overline{S_Y(f)} = \lim_{N \rightarrow \infty} \frac{1}{N\bar{l}} \overline{|Y_N(f)|^2} \tag{3.90}$$

where  $N\bar{l}$  is the average length of N pulses.

The square of the magnitude of the Fourier transform is

$$\begin{aligned}
|Y_N(f)|^2 &= \frac{1}{\omega_c^2} \sum_{k=1}^N \sum_{l=1}^N A_k A_l^* \\
&= \frac{1}{\omega_c^2} \left[ \sum_{n=1}^N |A_n|^2 + \sum_{k=1}^{N-1} \sum_{l=1}^{N-k} (A_{k+l} A_k^* + A_{k+l}^* A_k) \right] \\
&= \frac{1}{\omega_c^2} \left( \sum_{n=1}^N |A_n|^2 + 2\text{Re} \sum_{k=1}^{N-1} \sum_{l=1}^{N-k} A_{k+l} A_k^* \right)
\end{aligned} \tag{3.91}$$

since  $Z + Z^* = 2\text{Re}(Z)$ . Using Eq ( 3.89) the first summation in the above equation is

$$\begin{aligned}
\sum_{n=1}^N |A_n|^2 &= \sum_{n=1}^N \left\{ \left[ a_n e^{j(\beta_1 + \beta_2 + \dots + \beta_n)} (e^{-j\omega_c t_{n+1}} - e^{-j\omega_c t_n}) \right] \right. \\
&\quad \times \left. \left[ a_n e^{-j(\beta_1 + \beta_2 + \dots + \beta_n)} (e^{j\omega_c t_{n+1}} - e^{j\omega_c t_n}) \right] \right\} \\
&= \sum_{n=1}^N [a_n (e^{-j\omega_c t_{n+1}} - e^{-j\omega_c t_n}) a_n (e^{j\omega_c t_{n+1}} - e^{j\omega_c t_n})] \\
&= \sum_{n=1}^N a_n^2 [2 - e^{j\omega_c(t_{n+1} - t_n)} - e^{-j\omega_c(t_{n+1} - t_n)}] \tag{3.92}
\end{aligned}$$

The expected value of this summation is

$$E \left( \sum_{n=1}^N |A_n|^2 \right) = \sum_{n=1}^N E \left\{ a_n^2 [2 - e^{j\omega_c(t_{n+1} - t_n)} - e^{-j\omega_c(t_{n+1} - t_n)}] \right\} \tag{3.93}$$

$$= \sum_{n=1}^N E(a^2) \left\{ 2 - E [e^{j\omega_c(t_{n+1} - t_n)}] - E [e^{-j\omega_c(t_{n+1} - t_n)}] \right\} \tag{3.94}$$

Using Eq ( 3.82) this expectation becomes

$$E \left( \sum_{n=1}^N |A_n|^2 \right) = \sum_{n=1}^N E(a^2) (2 - \Omega - \Omega^*) \tag{3.95}$$

Since  $\Omega + \Omega^* = 2\text{Re}(\Omega)$

$$\begin{aligned}
E \left( \sum_{n=1}^N |A_n|^2 \right) &= N E(a^2) [2 - 2\text{Re}(\Omega)] \\
&= 2N E(a^2) [1 - \phi(f)] \tag{3.96}
\end{aligned}$$

Using Eq ( 3.89) the terms inside the second summation are



$$\begin{aligned}
A_{k+l}A_k^* &= \left[ a_{k+l}e^{j(\beta_1+\beta_2+\dots+\beta_{k+l})}(e^{-j\omega_c t_{k+l+1}} - e^{-j\omega_c t_{k+l}}) \right] \\
&\times \left[ a_k e^{-j(\beta_1+\beta_2+\dots+\beta_k)}(e^{j\omega_c t_{k+1}} - e^{j\omega_c t_k}) \right] \\
&= a_{k+l}a_k e^{j(\beta_{k+1}+\beta_{k+2}+\dots+\beta_{k+l})} \\
&\times \left[ e^{j\omega'(t_k-t_{k+l})} - e^{j\omega_c(t_k-t_{k+l+1})} - e^{j\omega_c(t_{k+1}-t_{k+l})} + e^{j\omega_c(t_{k+1}-t_{k+l+1})} \right] \quad (3.97)
\end{aligned}$$

The expected value of this term is

$$\begin{aligned}
E(A_{k+l}A_k^*) &= E(a_{k+l}a_k)E\left[e^{j(\beta_{k+1}+\beta_{k+2}+\dots+\beta_{k+l})}\right]\left\{E\left[e^{j\omega_c(t_k-t_{k+l})}\right]\right. \\
&\quad \left.-E\left[e^{j\omega_c(t_k-t_{k+l+1})}\right] - E\left[e^{j\omega_c(t_{k+1}-t_{k+l})}\right] + E\left[e^{j\omega_c(t_{k+1}-t_{k+l+1})}\right]\right\} \quad (3.98)
\end{aligned}$$

By using Eqs ( 3.74), ( 3.82), and ( 3.84) this expectation can be expressed

$$E(A_{k+l}A_k^*) = E^2(a)\Theta^l (\Omega^l - \Omega^{l+1} - \Omega^{l-1} + \Omega^l) \quad (3.99)$$

$$E(A_{k+l}A_k^*) = E^2(a) \left[ 2(\Omega\Theta)^l - \Omega(\Omega\Theta)^l - \frac{1}{\Omega}(\Omega\Theta)^l \right] \quad (3.100)$$

The summation of this expectation is

$$E\left(\sum_{k=1}^{N-1} \sum_{l=1}^{N-k} A_{k+l}A_k^*\right) = E^2(a) \sum_{k=1}^{N-1} \sum_{l=1}^{N-k} \left[ 2(\Omega\Theta)^l - \Omega(\Omega\Theta)^l - \frac{1}{\Omega}(\Omega\Theta)^l \right] \quad (3.101)$$

Let  $\gamma = (\Omega\Theta)$ . Now, the summation can be written

$$\begin{aligned}
\sum_{l=1}^{N-k} \gamma^l &= \sum_{m=0}^{N-k-1} \gamma^{m+1} \\
&= \gamma \sum_{m=0}^{N-k-1} \gamma^m \\
&= \frac{\gamma(1-\gamma^{N-k})}{1-\gamma}
\end{aligned} \tag{3.102}$$

The expectation of this summation now becomes

$$\begin{aligned}
E \left( \sum_{k=1}^{N-1} \sum_{l=1}^{N-k} A_{k+l} A_k^* \right) &= E^2(a) \sum_{k=1}^{N-1} \frac{\gamma(1-\gamma^{N-k})}{1-\gamma} \left( 2 - \Omega - \frac{1}{\Omega} \right) \\
&= E^2(a) \left[ \frac{\gamma}{1-\gamma} \left( 2 - \Omega - \frac{1}{\Omega} \right) \right] \sum_{k=1}^{N-1} (1-\gamma^{N-k}) \\
&= E^2(a) \left[ \frac{\gamma}{1-\gamma} \left( 2 - \Omega - \frac{1}{\Omega} \right) \right] \\
&\quad \times \left[ (N-1) - \frac{\gamma}{1-\gamma} (1-\gamma^{N-1}) \right] \\
&= E^2(a) \frac{\Theta}{1-\Omega\Theta} (-\Omega^2 + 2\Omega - 1) \\
&\quad \times \left[ N-1 - \frac{\gamma}{1-\gamma} (1-\gamma^{N-1}) \right] \\
&= -E^2(a) \frac{\Theta}{1-\Omega\Theta} (1-\Omega)^2 \left[ N-1 - \frac{\gamma}{1-\gamma} (1-\gamma^{N-1}) \right] \tag{3.103}
\end{aligned}$$

Substituting Eqs ( 3.96) and ( 3.103) into Eq ( 3.91) the square of the magnitude of the Fourier transform becomes

$$\begin{aligned}
|Y_N(f)|^2 &= \frac{1}{\omega_c^2} \left\{ \{ 2NE(a^2) [1-\phi(f)] \} \right. \\
&\quad \left. - 2E^2(a) Re \left\{ \frac{\Theta}{1-\Omega\Theta} (1-\Omega)^2 \left[ N-1 - \frac{\gamma}{1-\gamma} (1-\gamma^{N-1}) \right] \right\} \right\} \tag{3.104}
\end{aligned}$$

Using Eq ( 3.90) the power spectral density is obtained:

$$\overline{S_Y(f)} = \frac{1}{2\pi^2(f - f_o)^2} \left\{ E(a^2) [1 - \phi(f)] - \frac{1}{\pi} \text{Re} \left[ \frac{\Theta}{1 - \Omega\Theta} (1 - \Omega)^2 \right] \right\} \quad (3.105)$$

The above equation for the power spectral density can be quite useful in the given form, especially when the terms  $\Theta$  and  $\Omega$  are real quantities. However, this equation can be expanded as a function of the component parts of  $\Theta$  and  $\Omega$ . The coefficient of the  $E^2(a)$  term can be written

$$\begin{aligned} \text{Re} \left[ \frac{\Theta}{1 - \Omega\Theta} (1 - \Omega)^2 \right] &= \frac{\text{Re} [\Theta (1 - \Omega)^2 (1 - \Omega\Theta)^*]}{|1 - \Omega\Theta|^2} \\ &= \frac{N}{D} \end{aligned} \quad (3.106)$$

where

$N$  = the numerator

$D$  = the denominator

Using Eqs ( 3.77) and ( 3.78), the denominator in the above equation,  $D$ , can be written

$$\begin{aligned} D &= |1 - \Omega\Theta|^2 \\ &= (1 - \Omega\Theta)(1 - \Omega\Theta)^* \\ &= \{[1 - A\phi(f) + B\psi(f)] - j[A\psi(f) + B\phi(f)]\} \\ &\quad \times \{[1 - A\phi(f) + B\psi(f)] + j[A\psi(f) + B\phi(f)]\} \\ &= [1 - A\phi(f) + B\psi(f)]^2 + [A\psi(f) + B\phi(f)]^2 \end{aligned} \quad (3.107)$$

The components of the numerator are expanded here:

$$\begin{aligned}
 (1 - \Omega)^2 &= [1 - \phi(f) - j\psi(f)]^2 \\
 &= [1 - 2\phi(f) + \phi^2(f) - \psi^2(f)] - j2\psi(f)[1 - \phi(f)] \\
 \Theta(1 - \Omega)^2 &= (A + jB) \{ [1 - 2\phi(f) + \phi^2(f) - \psi^2(f)] - j2\psi(f)[1 - \phi(f)] \} \\
 &= \{ A [1 - 2\phi(f) + \phi^2(f) - \psi^2(f)] + 2B\psi(f)[1 - \phi(f)] \} \\
 &\quad + j \{ B [1 - 2\phi(f) + \phi^2(f) - \psi^2(f)] - 2A\psi(f)[1 - \phi(f)] \} \quad (3.108)
 \end{aligned}$$

$$\begin{aligned}
 (1 - \Omega\Theta)^* &= \{ 1 - [\phi(f) + j\psi(f)](A + jB) \}^* \\
 &= [1 - A\phi(f) + B\psi(f)] + j[A\psi(f) + B\phi(f)] \quad (3.109)
 \end{aligned}$$

Substituting Eqs ( 3.108) and ( 3.109), the expression for the numerator,  $N$ , can be written

$$\begin{aligned}
 N &= \text{Re} \left[ \Theta(1 - \Omega)^2 (1 - \Omega\Theta)^* \right] \\
 &= [1 - A\phi(f) + B\psi(f)] \{ A [1 - 2\phi(f) + \phi^2(f) - \psi^2(f)] + 2B\psi(f)[1 - \phi(f)] \} \\
 &\quad - [A\psi(f) + B\phi(f)] \{ B [1 - 2\phi(f) + \phi^2(f) - \psi^2(f)] - 2A\psi(f)[1 - \phi(f)] \} \\
 &= [1 - 2\phi(f) + \phi^2(f) - \psi^2(f)] [A - A^2\phi(f) - B^2\phi(f)] \\
 &\quad + [1 - \phi(f)] [2B\psi(f) + 2B^2\psi^2(f) + 2A^2\psi^2(f)] \\
 &= [1 - 2\phi(f) + \phi^2(f) - \psi^2(f)] \{ A - \phi(f) [A^2 + B^2] \} \\
 &\quad + 2\psi(f)[1 - \phi(f)] \{ B + \psi(f) [A^2 + B^2] \} \quad (3.110)
 \end{aligned}$$

Substituting Eqs ( 3.107) and ( 3.110) into Eq ( 3.106), and after some simplification, the power spectral density becomes

$$\overline{S_Y(f)} = \frac{1}{2l\pi^2(f-f_o)^2} \left\{ E(a^2)(1-\phi) - E^2(a) \right. \\ \left. \times \frac{(1-2\phi+\phi^2-\psi^2) [A-\phi(A^2+B^2)] + 2\psi(1-\phi) [B+\psi(A^2+B^2)]}{(1-A\phi+B\psi)^2 + (A\psi+B\phi)^2} \right\} \quad (3.111)$$

where the  $\phi$  and  $\psi$  terms are in general functions of frequency,  $f$ . The above equation does not reduce as elegantly as the equation presented in Lawson and Uhlenbeck (11:45). This is due to the introduction here of random amplitudes, while Lawson and Uhlenbeck only consider random phases and pulse spacing.

**3.3.2 Special Cases** The power spectral density presented can be a very useful tool to derive the power spectra of various waveforms. Two special cases of random waveforms are considered here. First, the PSD for the random binary transmission is derived. Finally, the power spectra for the random telegraph signal is developed.

**3.3.2.1 Random Binary Transmission** The special case of power spectral density of the random binary transmission is derived here from the general case of the PSD of a series of pulses with random amplitudes, spacing, and phases. Since the random binary transmission is a baseband process,  $f_o$  is set to zero. Two methods to achieve this PSD are illustrated here.

First, the amplitudes are assumed to have equally likely distributions as follows:

$$P(a = 1) = P(a = -1) = \frac{1}{2} \quad (3.112)$$

which is the same as

$$p(a) = \frac{1}{2} [\delta(a+1) + \delta(a-1)] \quad (3.113)$$

so that the expected values are given by

$$E(a) = 0 \quad (3.114)$$

$$E(a^2) = 1 \quad (3.115)$$

The phase will be assumed to be a constant value of zero. This is because the random amplitudes account for a phase change of either 0 or  $\pi$  radians (with an equally likely distribution). Substituting the above expectations into Eq (3.105)

$$S(f) = \frac{[1 - \phi(f)]}{2\bar{l}\pi^2 f^2} \quad (3.116)$$

For the random binary transmission, the pulse spacing is a constant value,  $T$ , so that

$$p(l) = \delta(l - T) \quad (3.117)$$

and  $\bar{l} = T$ . Next  $\phi(f)$  is derived from  $p(l)$  by use of Eq (3.78) as follows:

$$\begin{aligned} \Omega &= \phi(f) + j\psi(f) \\ &= \int_0^\infty p(l)e^{-j2\pi lf} dl \\ &= \int_0^\infty p(l) = \delta(l - T)e^{-j2\pi lf} dl \\ &= e^{-j2\pi fT} \\ &= \cos 2\pi fT + j \sin 2\pi fT \end{aligned} \quad (3.118)$$

Now,  $\phi(f) = \cos 2\pi fT$ , and the PSD is now

$$\begin{aligned}
 S(f) &= \frac{(1 - \cos 2\pi fT)}{2T\pi^2 f^2} \\
 &= \frac{2\sin^2 \pi fT}{2T\pi^2 f^2} \\
 &= T \left( \frac{\sin \pi fT}{\pi fT} \right)^2
 \end{aligned} \tag{3.119}$$

which is of the same form as the PSD of the binary random transmission presented by Papoulis (14:341). This power spectral density is illustrated in Figure 3.3.

A slightly different method will be demonstrated, which yields the same result. In this case, take the amplitudes to be a constant value, so that

$$E(a) = E(a^2) = 1 \tag{3.120}$$

and let the phase changes be either 0 or  $\pi$  radians, with equal likelihood, so that

$$p(\alpha) = \frac{1}{2} \left[ \delta(\alpha) + \delta\left(\alpha - \frac{1}{2}\right) \right] \tag{3.121}$$

From Eq (3.77) the value of  $\Theta$  is derived:

$$\begin{aligned}
 \Theta &= \int_{-\frac{1}{2}}^{\frac{1}{2}} p(\alpha) e^{j2\pi\alpha} d\alpha \\
 &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{2} \left[ \delta(\alpha) + \delta\left(\alpha - \frac{1}{2}\right) \right] e^{j2\pi\alpha} d\alpha \\
 &= 0
 \end{aligned} \tag{3.122}$$

and  $A = B = 0$ , so both Eq ( 3.105) and Eq ( 3.111) reduce to

$$S(f) = \frac{[1 - \phi(f)]}{2\bar{l}\pi^2 f^2} \quad (3.123)$$

Since the probability density of the pulse spacing,  $p(l)$ , has not changed, the PSD reduces to the form of Eq ( 3.119).

**3.3.2.2 Random Telegraph Signal** In this section the PSD of the random telegraph signal is derived. This waveform is defined by Papoulis as a point process with an underlying Poisson distribution (14:288-290). This process is a baseband process, so the frequency  $f_o$  will be set to zero. The interarrival times of this point process will determine the pulse length,  $l$ , for the case considered here. The interarrival times of a Poisson process are shown by Davenport to have an exponential density (3:471). Therefore, in this case the probability density,  $p(l)$ , is given by

$$p(l) = \lambda e^{-\lambda l}, l \geq 0 \quad (3.124)$$

For the random telegraph signal, the pulse amplitudes transition from +1 to -1, and vice versa, at each interarrival time. Therefore, one way to view the distribution of the amplitudes and phases is to set

$$E(a) = E(a^2) = 1 \quad (3.125)$$

and the density of  $\alpha$  to

$$p(\alpha) = \delta\left(\alpha - \frac{1}{2}\right) \quad (3.126)$$

From Eq ( 3.77) this leads to



$$\begin{aligned}
\Theta &= A + jB \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} p(\alpha) e^{j2\pi\alpha} d\alpha \\
&= e^{j\pi} \\
&= -1
\end{aligned} \tag{3.127}$$

so that  $A = -1$  and  $B = 0$ . Substitution of the above expressions into Eq ( 3.111) yields the power spectral density of the signal:

$$\begin{aligned}
S(f) &= \frac{1}{2\bar{l}\pi^2 f^2} \left[ (1 - \phi) - \frac{(1 - 2\phi + \phi^2 - \psi^2)(-1 - \phi) + 2\psi^2(1 - \phi)}{(1 + \phi)^2 + \psi^2} \right] \\
&= \frac{1}{\bar{l}\pi^2 f^2} \left[ \frac{1 - \phi^2 - \psi^2}{(1 + \phi)^2 + \psi^2} \right]
\end{aligned} \tag{3.128}$$

From Eq ( 3.124), the following expectation results:

$$\begin{aligned}
\bar{l} &= \int_0^{\infty} l \lambda e^{-\lambda l} dl \\
&= \frac{1}{\lambda}
\end{aligned} \tag{3.129}$$

And, from Eq ( 3.78)

$$\begin{aligned}
\Omega &= \phi + j\psi \\
&= \int_0^{\infty} p(l) e^{-j2\pi f l} dl
\end{aligned}$$

$$\begin{aligned}
\Omega &= \int_0^{\infty} \lambda e^{-l(j2\pi f + \lambda)} dl \\
&= \frac{\lambda}{\lambda + j\omega}
\end{aligned} \tag{3.130}$$

where  $\omega = 2\pi f$ .  $\Omega$  can be broken onto its component parts as follows:

$$\Omega = \frac{\lambda^2}{\lambda^2 + \omega^2} - j \frac{\lambda\omega}{\lambda^2 + \omega^2} \tag{3.131}$$

Substituting the above into Eq ( 3.128), the PSD is obtained:

$$\begin{aligned}
S(\omega) &= \frac{4\lambda}{\omega^2} \left[ \frac{1 - \frac{\lambda^4}{(\lambda^2 + \omega^2)^2} - \frac{\lambda^2\omega^2}{(\lambda^2 + \omega^2)^2}}{\left(1 + \frac{\lambda^2}{\lambda^2 + \omega^2}\right)^2 + \frac{\lambda^2\omega^2}{(\lambda^2 + \omega^2)^2}} \right] \\
&= \frac{4\lambda}{\omega^2} \left[ \frac{(\lambda^2 + \omega^2)^2 - \lambda^4 - \lambda^2\omega^2}{(2\lambda^2 + \omega^2)^2 + \lambda^2\omega^2} \right] \\
&= \frac{4\lambda}{\omega^2} \left[ \frac{\omega^4 + \lambda^2\omega^2}{4\lambda^4 + 5\lambda^2\omega^2 + \omega^4} \right] \\
&= \frac{4\lambda}{4\lambda^2 + \omega^2} \\
S(f) &= \frac{4\lambda}{4\lambda^2 + (2\pi f)^2}
\end{aligned} \tag{3.132}$$

This equation for the power spectral density is illustrated in Figure 3.5.

This expression is the same as that given by Papoulis for the power spectral density of the random telegraph signal (14:341).

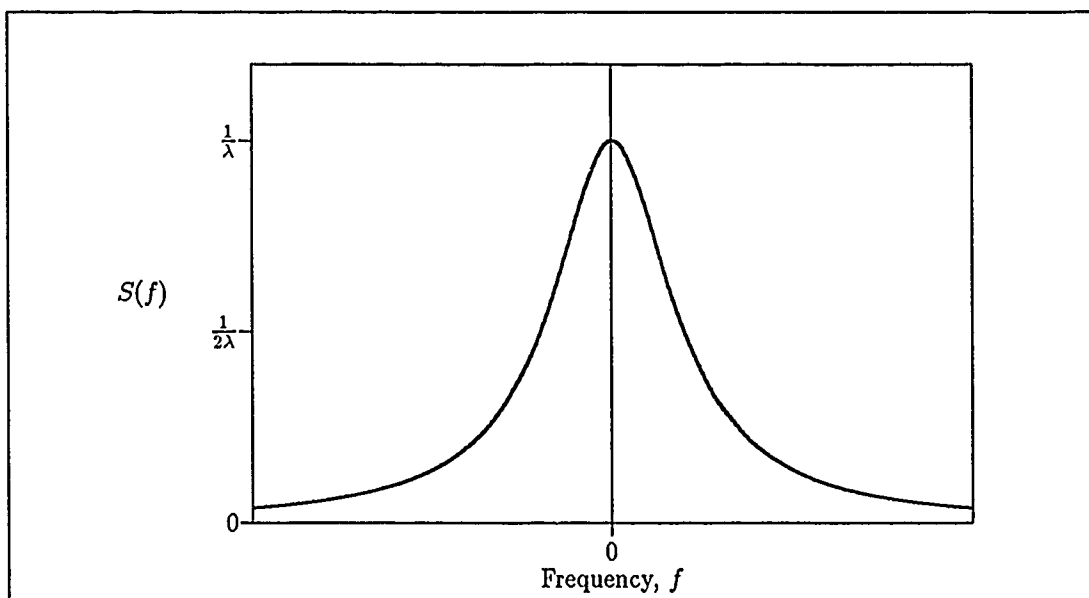


Figure 3.5. Random Telegraph Signal PSD

#### IV. Probability Densities and Expectations

The probability density functions of the noise waveform and the signal plus noise waveform, along with the power spectral density, are one of the most important properties required in the analysis of the effects of noise jamming on radar receivers. The PDF describes the distribution of the amplitude of the noise jamming waveform. In this chapter the probability density functions of the amplitude of the output of both a linear and quadratic detector are derived. These PDF's are derived for the case of a noise only input as well as the case of the input of signal plus noise. Lastly, several expected values are derived. The mean and mean square values are determined for the two cases: noise only and signal plus noise. Also, the autocorrelation function for the signal plus noise case is derived for both detector types.

##### 4.1 Probability Densities of the Output of a Linear Detector

In this section, the probability density functions for the output of a linear (envelope) detector are developed. The PDF's will be derived for the case of the input of noise only as well as the case of a signal plus noise.

*4.1.1 Probability Density for Noise Only* The noise analyzed here is assumed to be white and have a Gaussian distribution with mean value of zero and a variance of  $\sigma^2$ . This noise,  $n_t(t)$ , is input to the IF filter, whose transfer function is  $H(f)$ . The noise output from the IF filter,  $n(t)$ , can be portrayed in quadrature form as

$$n(t) = n_x(t) \cos 2\pi f_c t + n_y(t) \sin 2\pi f_c t \quad (4.1)$$

where

$f_c$  = the center frequency of the IF filter passband

The noise components  $n_x(t)$  and  $n_y(t)$  are independent, zero mean Gaussian random variables with PDF's

$$p(n_x) = \frac{1}{\sqrt{2\pi W}} \exp\left(-\frac{n_x^2}{2W}\right) \quad (4.2)$$

$$p(n_y) = \frac{1}{\sqrt{2\pi W}} \exp\left(-\frac{n_y^2}{2W}\right) \quad (4.3)$$

where

$$W = \sigma^2 \int_{-\infty}^{\infty} |H(f)|^2 df \quad (4.4)$$

Because these two noise components are statistically independent, their joint PDF,  $p(n_x, n_y)$ , is

$$p(n_x, n_y) = \frac{1}{2\pi W} \exp\left(-\frac{n_x^2 + n_y^2}{2W}\right) \quad (4.5)$$

The output of the linear detector,  $L(t)$ , is the envelope of the input signal, defined by the expression

$$L_n(t) = \sqrt{n_x^2(t) + n_y^2(t)} \quad (4.6)$$

where the subscript  $n$  is to denote the noise only condition. The terms  $n_x$  and  $n_y$  can be transformed to polar form in terms of  $L_n$  and  $\phi_n$  where

$$n_x = L_n \cos \phi_n = h_1(L_n, \phi_n) \quad (4.7)$$

$$n_y = L_n \sin \phi_n = h_2(L_n, \phi_n) \quad (4.8)$$

The Jacobian of this transformation is

$$J = \begin{vmatrix} \frac{\partial h_1}{\partial L_n} & \frac{\partial h_1}{\partial \phi_n} \\ \frac{\partial h_2}{\partial L_n} & \frac{\partial h_2}{\partial \phi_n} \end{vmatrix} \quad (4.9)$$

which reduces to

$$\begin{aligned} J &= L_n \cos^2 \phi_n + L_n \sin^2 \phi_n \\ &= L_n \end{aligned} \quad (4.10)$$

The joint density of  $L_n$  and  $\phi_n$  can be derived from the joint density of  $n_x$  and  $n_y$  by the equation

$$\begin{aligned} p(L_n, \phi_n) &= p_{n_x, n_y} [h_1(L_n, \phi_n), h_2(L_n, \phi_n)] J \\ &= \frac{1}{2\pi W} \exp \left( -\frac{L_n^2 \cos^2 \phi_n + L_n^2 \sin^2 \phi_n}{2W} \right) L_n \\ &= \frac{L_n}{2\pi W} \exp \left( -\frac{L_n^2}{2W} \right) \end{aligned} \quad (4.11)$$

From the joint PDF, the marginal PDF of the envelope,  $p(L_n)$ , can be obtained by integrating over all values of  $\phi_n$  so that

$$\begin{aligned} p(L_n) &= \int_0^{2\pi} p(L_n, \phi_n) d\phi_n \\ &= \int_0^{2\pi} \frac{L_n}{2\pi W} \exp \left( -\frac{L_n^2}{2W} \right) d\phi_n \\ &= \frac{L_n}{W} \exp \left( -\frac{L_n^2}{2W} \right) \end{aligned} \quad (4.12)$$

for all values of  $L_n \geq 0$ . This probability density is known as the Rayleigh PDF and is illustrated in Figure 4.1.

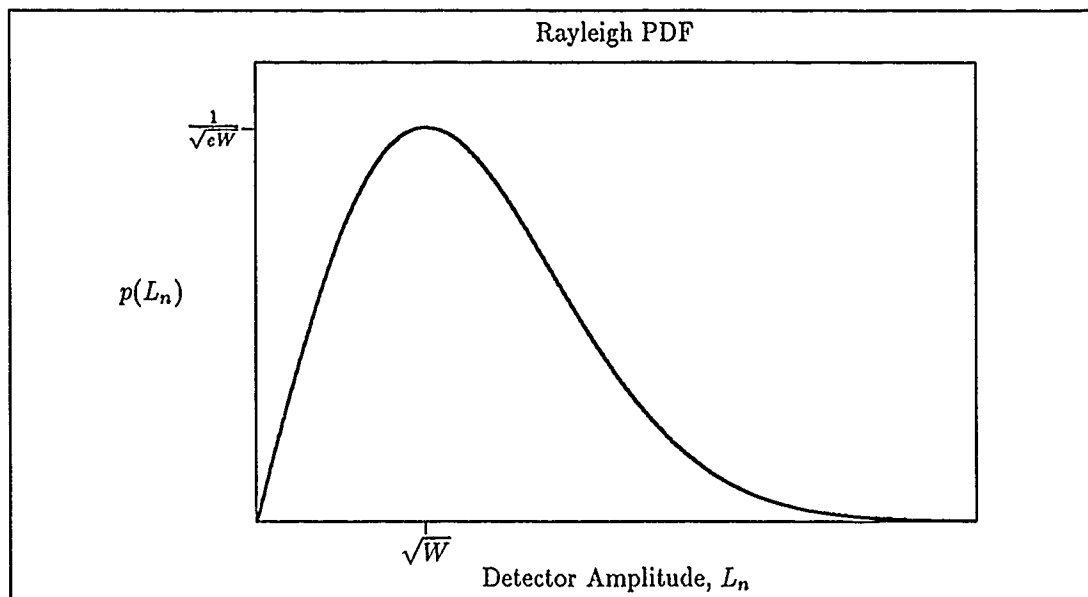


Figure 4.1. PDF of Noise Only for a Linear Detector

It can be shown in a similar manner that the phase term,  $\phi_n$ , is uniformly distributed over the interval  $(0, 2\pi)$ , and that  $L_n$  and  $\phi_n$  are statistically independent random variables.

**4.1.2 Probability Density for Signal Plus Noise** The PDF for a signal plus Gaussian noise will now be derived for the case of a linear detector. It is assumed that the signal and noise are added together prior to the detector, and that the signal is

$$s(t) = \alpha(t) \cos 2\pi f_c t + \beta(t) \sin 2\pi f_c t \quad (4.13)$$

so that from Eqs ( 4.1) and ( 4.13) the input to the detector,  $e(t)$ , is

$$\begin{aligned}
e(t) &= s(t) + n(t) \\
&= [\alpha(t) + n_x(t)] \cos 2\pi f_c t + [\beta(t) + n_y(t)] \sin 2\pi f_c t \\
&= x(t) \cos 2\pi f_c t + y(t) \sin 2\pi f_c t
\end{aligned} \tag{4.14}$$

where  $x(t)$  and  $y(t)$  are independent Gaussian random variables. Since  $n_x(t)$  and  $n_y(t)$  are zero mean random variables, it follows that the means of  $x(t)$  and  $y(t)$  are now

$$E[x(t)] = \alpha(t) \tag{4.15}$$

$$E[y(t)] = \beta(t) \tag{4.16}$$

and the variance of each is  $W$ , as was the case for noise only. Now, the joint PDF of  $x$  and  $y$  can be written:

$$p(x, y) = \frac{1}{2\pi W} \exp \left[ -\frac{(x - \alpha)^2 + (y - \beta)^2}{2W} \right] \tag{4.17}$$

This density can be transformed into polar form in terms of  $L_{s+n}$  and  $\phi_{s+n}$ , as was done for the case of noise only, using the equation

$$p(L_{s+n}, \phi_{s+n}) = p_{x,y} [h_1(L_{s+n}, \phi_{s+n}), h_2(L_{s+n}, \phi_{s+n})] J \tag{4.18}$$



where

$$h_1(L_{s+n}, \phi_{s+n}) = L_{s+n} \cos \phi_{s+n}$$

$$h_2(L_{s+n}, \phi_{s+n}) = L_{s+n} \sin \phi_{s+n}$$

$$J = L_{s+n}$$

This leads to the joint density

$$\begin{aligned} p(L_{s+n}, \phi_{s+n}) &= \frac{L_{s+n}}{2\pi W} \exp \left[ -\frac{(L_{s+n} \cos \phi_{s+n} - \alpha)^2 + (L_{s+n} \sin \phi_{s+n} - \beta)^2}{2W} \right] \\ &= \frac{L_{s+n}}{2\pi W} \exp \left[ -\frac{L_{s+n}^2 - 2L_{s+n}(\alpha \cos \phi_{s+n} + \beta \sin \phi_{s+n}) + \alpha^2 + \beta^2}{2W} \right] \end{aligned} \quad (4.19)$$

To arrive at the PDF of the amplitude of the detector output,  $p(L_{s+n})$ , the above joint PDF is integrated relative to  $\phi_{s+n}$  over the interval  $(0, 2\pi)$ :

$$\begin{aligned} p(L_{s+n}) &= \int_0^{2\pi} p(L_{s+n}, \phi_{s+n}) d\phi_{s+n} \\ &= \int_0^{2\pi} \frac{L_{s+n}}{2\pi W} \exp \left[ -\frac{L_{s+n}^2 - 2L_{s+n}(\alpha \cos \phi_{s+n} + \beta \sin \phi_{s+n}) + \alpha^2 + \beta^2}{2W} \right] d\phi_{s+n} \\ &= \frac{L_{s+n}}{2\pi W} \exp \left( -\frac{L_{s+n}^2 + \alpha^2 + \beta^2}{2W} \right) \\ &\quad \times \int_0^{2\pi} \exp \left[ -\frac{L_{s+n}}{W} (\alpha \cos \phi_{s+n} + \beta \sin \phi_{s+n}) \right] d\phi_{s+n} \end{aligned} \quad (4.20)$$

The integral in the above equation is of the form of Eq (3.937,2) found in Gradshteyn and Ryzhik, and reduces to (8:488)

$$\int_0^{2\pi} \exp \left[ -\frac{L_{s+n}}{W} (\alpha \cos \phi_{s+n} + \beta \sin \phi_{s+n}) \right] d\phi_{s+n} = 2\pi I_0 \left( \frac{L_{s+n}}{W} \sqrt{\alpha^2 + \beta^2} \right) \quad (4.21)$$

and upon substitution of the above equation into Eq ( 4.20), the PDF of the envelope now becomes

$$\begin{aligned} p(L_{s+n}) &= \frac{L_{s+n}}{2\pi W} \exp\left(-\frac{L_{s+n}^2 + \alpha^2 + \beta^2}{2W}\right) \left[2\pi I_0\left(\frac{L_{s+n}}{W} \sqrt{\alpha^2 + \beta^2}\right)\right] \\ &= \frac{L_{s+n}}{W} \exp\left(-\frac{L_{s+n}^2 + \alpha^2 + \beta^2}{2W}\right) I_0\left(\frac{L_{s+n}}{W} \sqrt{\alpha^2 + \beta^2}\right) \end{aligned} \quad (4.22)$$

for  $L_{s+n} \geq 0$ . The term  $I_0$  in the above equation is the modified Bessel function of order zero. This expression for the probability density is known as the Rician PDF. The Rician PDF is illustrated in Figure 4.2, similar to the form of a related plot found in Lawson and Uhlenbeck (11:154). Two values of  $z = (\alpha^2 + \beta^2)/2W$  are plotted in Figure 4.2.

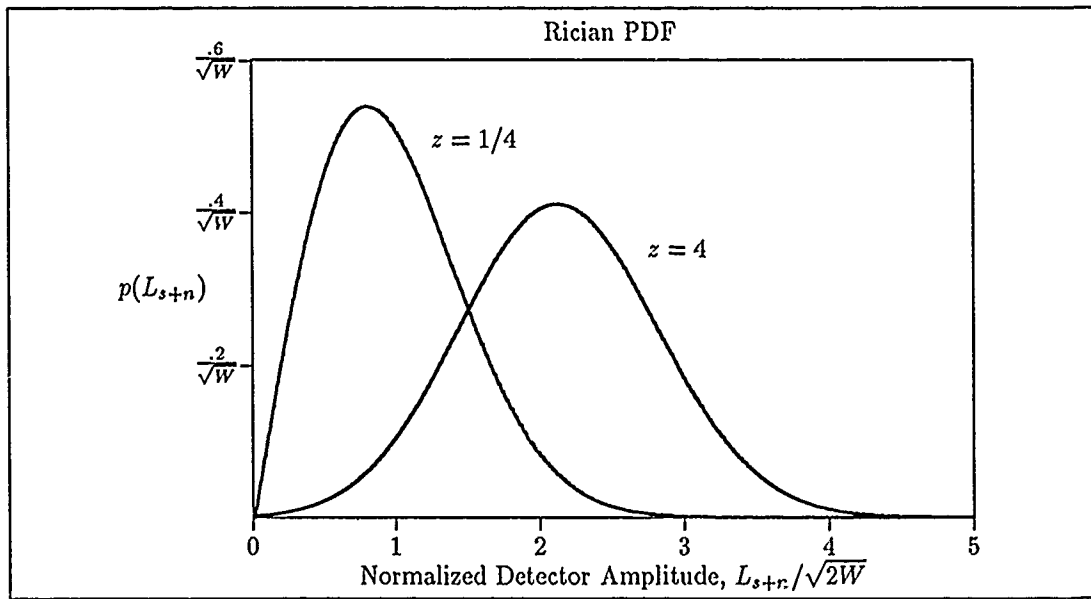


Figure 4.2. PDF of Signal Plus Noise for a Linear Detector

DiFranco and Rubin provide useful approximations for two cases of the modified Bessel function. For the small argument case (5:344)

$$I_0(x) = 1 + \frac{x^2}{4} + \frac{x^4}{64} + \dots, x < 1 \quad (4.23)$$

and for the large argument case (5:344)

$$I_0(x) = \frac{e^x}{\sqrt{2\pi x}} \left( 1 + \frac{1}{8x} \dots \right), x > 1 \quad (4.24)$$

**4.1.3 Expected Values** The expected values of  $L$  and  $L^2$  for the noise only and signal plus noise cases are derived here. Also, the autocorrelation function,  $\overline{L_1 L_2}$  for the signal plus noise case is approximated. First, the case of the average value of the envelope of the noise only output is considered. With the PDF of  $L_n$  given by Eq (4.12), the average value of the noise amplitude is given by

$$\begin{aligned} E(L_n) &= \int_{-\infty}^{\infty} L_n p(L_n) dL_n \\ &= \int_0^{\infty} \frac{L_n^2}{W} \exp\left(-\frac{L_n^2}{2W}\right) dL_n \\ &= 2a \int_0^{\infty} L_n^2 e^{-aL_n^2} dL_n \end{aligned} \quad (4.25)$$

where

$$a = \frac{1}{2W}$$

The above integral is of the form of Eq (666) in the *CRC Standard Mathematical Tables* (16:465).

Using this relationship, the expectation becomes

$$E(L_n) = \sqrt{\frac{\pi W}{2}} \quad (4.26)$$

Next, the expected value of  $L_n^2$  is calculated:

$$\begin{aligned}
E(L_n^2) &= \int_{-\infty}^{\infty} L_n^2 p(L_n) dL_n \\
&= \int_0^{\infty} \frac{L_n^3}{W} \exp\left(-\frac{L_n^2}{2W}\right) dL_n \\
&= \frac{1}{2W} \int_0^{\infty} a \exp\left(-\frac{a}{2W}\right) da
\end{aligned} \tag{4.27}$$

where

$$\begin{aligned}
a &= L_n^2 \\
da &= 2L_n dL_n
\end{aligned}$$

This integral is of the form of Eq (661) in the *CRC Tables* (16:465). After substitution, the expectation becomes

$$\begin{aligned}
E(L_n^2) &= \frac{1}{2W} \left[ \left( \frac{1}{2W} \right)^2 \right]^{-1} \\
&= 2W
\end{aligned} \tag{4.28}$$

Now, the expectations for the signal plus noise case are derived. With the PDF of the signal plus noise output given by Eq ( 4.22), the expectation of  $L_{s+n}$  is

$$\begin{aligned}
E(L_{s+n}) &= \int_{-\infty}^{\infty} L_{s+n} p(L_{s+n}) dL_{s+n} \\
&= \int_0^{\infty} \frac{L_{s+n}^2}{W} \exp\left(-\frac{L_{s+n}^2 + \alpha^2 + \beta^2}{2W}\right) I_0\left(\frac{L_{s+n}}{W} \sqrt{\alpha^2 + \beta^2}\right) dL_{s+n}
\end{aligned} \tag{4.29}$$

The integral in the above equation is defined in Lawson and Uhlenbeck by the following expression (11:174):

$$\int_0^\infty t^{\mu-1} I_\nu(at) e^{-p^2 t^2} dt = \frac{\Gamma\left(\frac{\mu+\nu}{2}\right) \left(\frac{a}{2p}\right)^\nu}{2p^\mu \Gamma(\nu+1)} \exp\left(\frac{a^2}{4p^2}\right) {}_1F_1\left(\frac{\nu-\mu}{2}+1, \nu+1; -\frac{a^2}{4p^2}\right) \quad (4.30)$$

For the case of  $\nu = 0$ , the above equation becomes

$$\int_0^\infty t^{\mu-1} I_0(at) e^{-p^2 t^2} dt = \frac{\Gamma\left(\frac{\mu}{2}\right)}{2p^\mu} \exp\left(\frac{a^2}{4p^2}\right) {}_1F_1\left(1-\frac{\mu}{2}, 1; -\frac{a^2}{4p^2}\right) \quad (4.31)$$

where  ${}_1F_1(a, b; z)$  is the confluent hypergeometric function, defined by (11:174)

$${}_1F_1(a, b; z) = 1 + \frac{a}{b}z + \frac{a(a+1)}{b(b+1)} \frac{z^2}{2!} + \frac{a(a+1)(a+2)}{b(b+1)(b+2)} \frac{z^3}{3!} + \dots \quad (4.32)$$

Using the integral relationship of Eq (4.31), the expectation,  $E(L_{s+n})$ , becomes

$$E(L_{s+n}) = \frac{1}{W} \exp\left(-\frac{\alpha^2 + \beta^2}{2W}\right) \left[ \frac{\Gamma\left(\frac{3}{2}\right) (2W)^{\frac{3}{2}}}{2} \exp\left(\frac{\alpha^2 + \beta^2}{2W}\right) {}_1F_1\left(-\frac{1}{2}, 1; -\frac{\alpha^2 + \beta^2}{2W}\right) \right] \quad (4.33)$$

From the *CRC Tables* Eq (605),  $\Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$  (16:460). With this relationship, the expectation can be reduced to the following expression:

$$E(L_{s+n}) = \sqrt{\frac{\pi W}{2}} {}_1F_1\left(-\frac{1}{2}, 1; -\frac{\alpha^2 + \beta^2}{2W}\right) \quad (4.34)$$

Lawson and Uhlenbeck provide an asymptotic approximation for  ${}_1F_1(a, b; z)$  for large negative values of  $z$  (11:174). This approximation is

$${}_1F_1(a, b; z) \approx \frac{\Gamma(b)}{\Gamma(b-a)} (-z)^{-a} \left[ 1 - \frac{a(a-b+1)}{z} + \frac{a(a+1)(a-b+1)(a-b+2)}{z^2} + \dots \right] \quad (4.35)$$

In the above case for  $E(L_{s+n})$ , where  $z = -\frac{\alpha^2 + \beta^2}{2W}$ , the confluent hypergeometric function becomes

$${}_1F_1\left(-\frac{1}{2}, 1; z\right) \approx \frac{\Gamma(1)}{\Gamma\left(\frac{3}{2}\right)} (-z)^{\frac{1}{2}} \left[ 1 - \frac{1}{4z} + \frac{1}{16z^2} + \dots \right] \quad (4.36)$$

By keeping only the first term of the expansion, and substituting into Eq ( 4.34) the average value of the detector output is approximated by

$$\begin{aligned} E(L_{s+n}) &\approx \sqrt{\frac{\pi W}{2}} \frac{\Gamma(1)}{\Gamma\left(\frac{3}{2}\right)} \sqrt{\frac{\alpha^2 + \beta^2}{2W}} \\ &\approx \sqrt{\frac{\pi W}{2}} \frac{2}{\sqrt{\pi}} \sqrt{\frac{\alpha^2 + \beta^2}{2W}} \\ &\approx \sqrt{\alpha^2 + \beta^2} \end{aligned} \quad (4.37)$$

Now, the expected value of  $L_{s+n}^2$  is derived. Using Eq ( 4.22) the expectation is written

$$\begin{aligned} E(L_{s+n}^2) &= \int_{-\infty}^{\infty} L_{s+n}^2 p(L_{s+n}) dL_{s+n} \\ &= \int_0^{\infty} \frac{L_{s+n}^3}{W} \exp\left(-\frac{L_{s+n}^2 + \alpha^2 + \beta^2}{2W}\right) I_0\left(\frac{L_{s+n}}{W} \sqrt{\alpha^2 + \beta^2}\right) dL_{s+n} \end{aligned} \quad (4.38)$$

From Eq ( 4.31) the mean squared value becomes

$$\begin{aligned}
E(L_{s+n}^2) &= \frac{1}{W} \exp\left(-\frac{\alpha^2 + \beta^2}{2W}\right) \left[ \frac{\Gamma(2)4W^2}{2} \exp\left(\frac{\alpha^2 + \beta^2}{2W}\right) {}_1F_1\left(-1, 1; -\frac{\alpha^2 + \beta^2}{2W}\right) \right] \\
&= 2W {}_1F_1\left(-1, 1; -\frac{\alpha^2 + \beta^2}{2W}\right)
\end{aligned} \tag{4.39}$$

It follows from Eq ( 4.32) that the confluent hypergeometric function in the above equation reduces to

$${}_1F_1(-1, 1; z) = 1 - z \tag{4.40}$$

Therefore, the mean square value becomes

$$\begin{aligned}
E(L_{s+n}^2) &= 2W \left(1 + \frac{\alpha^2 + \beta^2}{2W}\right) \\
&= 2W + \alpha^2 + \beta^2
\end{aligned} \tag{4.41}$$

The last expectation considered for the linear detector output is the autocorrelation function of the amplitude for the case of signal plus noise. First, the joint PDF of  $x_1$ ,  $y_1$ ,  $x_2$  and  $y_2$  is required. The subscripts refer to the times  $t_1$  and  $t_2$  respectively. This PDF is a four-variate Gaussian of the general form

$$p(\mathbf{X}) = \left[ (2\pi)^{\frac{n}{2}} |\mathbf{V}|^{\frac{1}{2}} \right]^{-1} \exp \left[ -\frac{1}{2} (\mathbf{X} - \mathbf{m})^T \mathbf{V}^{-1} (\mathbf{X} - \mathbf{m}) \right] \tag{4.42}$$

where

$$n = 4$$

$$\mathbf{X} = \begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{bmatrix}; \mathbf{m} = \begin{bmatrix} \alpha_1 \\ \beta_1 \\ \alpha_2 \\ \beta_2 \end{bmatrix}$$

$\mathbf{V}$  is the covariance matrix defined by

$$\mathbf{V} = W \begin{bmatrix} 1 & 0 & \rho & 0 \\ 0 & 1 & 0 & \rho \\ \rho & 0 & 1 & 0 \\ 0 & \rho & 0 & 1 \end{bmatrix} \quad (4.43)$$

which follows from the fact that the  $x_i$  and  $y_i$  terms are uncorrelated, but  $x_1$  and  $x_2$  are not uncorrelated. Likewise for  $y_1$  and  $y_2$ . The term,  $\rho$ , is the normalized correlation coefficient defined by

$$\rho = \frac{\text{cov}(x_i, x_j)}{\sigma(x_i)\sigma(x_j)} \quad (4.44)$$

As Lawson and Uhlenbeck point out, the equation for the normalized correlation coefficient in this case can be written in terms of the IF filter transfer function,  $H(f)$ , as follows (11:155):

$$\rho(\tau) = \frac{\int_{-\infty}^{\infty} |H(f)|^2 \cos 2\pi f \tau df}{\int_{-\infty}^{\infty} |H(f)|^2 df} \quad (4.45)$$

where

$$\tau = t_2 - t_1$$



The above leads to the following equation for the joint PDF of  $x_1, y_1, x_2$  and  $y_2$ :

$$p(x_1, y_1; x_2, y_2) = \frac{1}{4\pi^2 W^2 (1 - \rho^2)} \exp \left\{ -\frac{1}{2W(1 - \rho^2)} \left\{ (x_1 - \alpha_1)^2 + (y_1 - \beta_1)^2 + (x_2 - \alpha_2)^2 + (y_2 - \beta_2)^2 - 2\rho [(x_1 - \alpha_1)(x_2 - \alpha_2) + (y_1 - \beta_1)(y_2 - \beta_2)] \right\} \right\} \quad (4.46)$$

Now the autocorrelation function can be derived:

$$\begin{aligned} \overline{L_1 L_2} &= E \left[ \left( \sqrt{x_1^2 + y_1^2} \right) \left( \sqrt{x_2^2 + y_2^2} \right) \right] \\ &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left( \sqrt{x_1^2 + y_1^2} \right) \left( \sqrt{x_2^2 + y_2^2} \right) p(x_1, y_1; x_2, y_2) dx_1 dy_1 dx_2 dy_2 \quad (4.47) \end{aligned}$$

The integrals in the above equation can not be solved exactly. However, Lawson and Uhlenbeck provide two approximations for these integrals: one for the small signal-to-noise ratio case, and one for the large signal-to-noise ratio case (11:156-157). For the small signal-to-noise ratio case the autocorrelation is approximately

$$\begin{aligned} \overline{L_1 L_2} &\approx \frac{\pi}{32W} [(\alpha_1^2 + \beta_1^2 + 4W)(\alpha_2^2 + \beta_2^2 + 4W) \\ &\quad + 4W\rho(\alpha_1\alpha_2 + \beta_1\beta_2) + 4W^2\rho^2] \quad (4.48) \end{aligned}$$

and for a large signal-to-noise ratio the autocorrelation is approximated by

$$\begin{aligned} \overline{L_1 L_2} \approx & [(\alpha_1^2 + \beta_1^2)(\alpha_2^2 + \beta_2^2)]^{\frac{1}{2}} + W\rho \frac{\alpha_1 \alpha_2 + \beta_1 \beta_2}{[(\alpha_1^2 + \beta_1^2)(\alpha_2^2 + \beta_2^2)]^{\frac{1}{2}}} \\ & + \frac{W^2 \rho^2}{2} \frac{(\alpha_1 \alpha_2 + \beta_1 \beta_2)^2}{[(\alpha_1^2 + \beta_1^2)(\alpha_2^2 + \beta_2^2)]^{\frac{3}{2}}} \end{aligned} \quad (4.49)$$

#### 4.2 Probability Densities of the Output of a Quadratic Detector

The PDF's of the output of a quadratic (square-law) detector are developed in this section. As was the case for the linear detector, the PDF's for both noise only and signal plus noise are presented here. The output of the quadratic detector,  $Q(t)$ , can be thought of as merely the square of the output of the linear detector. For this case, Papoulis gives an expression for the PDF of the output of a square-law detector for  $y = ax^2$ :

$$p_y(y) = \frac{1}{2\sqrt{ay}} \left[ p_x\left(\sqrt{\frac{y}{a}}\right) + p_x\left(-\sqrt{\frac{y}{a}}\right) \right] U(y) \quad (4.50)$$

where

$p_x(x)$  = the PDF of the random variable  $x$

$U(y)$  = the unit step function

Since the PDF's  $p(L_n)$  and  $p(L_{s+n})$  are non zero only for  $L > 0$ , the PDF of the output of the quadratic detector,  $Q(t) = L^2(t)$ , can be written

$$p(Q) = \frac{1}{2\sqrt{Q}} p_L\left(\sqrt{Q}\right) \quad (4.51)$$

This result will be used to derive the PDF's for the quadratic detector. As before the signal out of the IF amplifier is given by Eq ( 4.13) and the noise process is given by Eq ( 4.1).

For the case of noise only, the PDF for the linear detector was given by Eq ( 4.12). After applying Eq ( 4.51), the PDF for the output of the quadratic detector for an input of Gaussian noise only is

$$\begin{aligned}
 p(Q_n) &= \frac{1}{2\sqrt{Q_n}} p_{L_n}(\sqrt{Q_n}) \\
 &= \left( \frac{1}{2\sqrt{Q_n}} \right) \frac{Q_n}{W} \exp\left(-\frac{Q_n}{2W}\right) \\
 &= \frac{1}{2W} \exp\left(-\frac{Q_n}{2W}\right)
 \end{aligned} \tag{4.52}$$

which is only non zero for  $Q_n > 0$ . Therefore, the output of the quadratic detector has an exponential probability density. This PDF is illustrated by Figure 4.3.

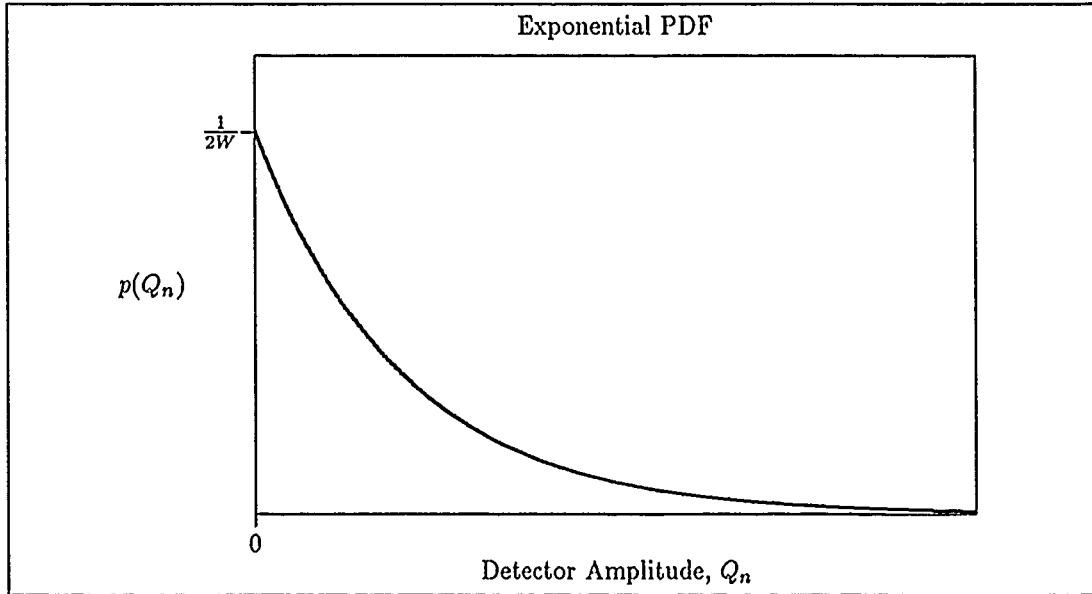


Figure 4.3. PDF of Noise Only for the Quadratic Detector

A similar approach is taken for the signal plus noise case. By applying Eq ( 4.51) to the PDF of the signal plus noise given by Eq ( 4.22) the PDF of the signal plus noise output of the quadratic

detector  $p(Q_{s+n})$  is obtained:

$$\begin{aligned}
 p(Q_{s+n}) &= \frac{1}{2\sqrt{Q_{s+n}}} p_{L_{s+n}}(\sqrt{Q_{s+n}}) \\
 &= \left( \frac{1}{2\sqrt{Q_{s+n}}} \right) \frac{\sqrt{Q_{s+n}}}{W} \exp\left(-\frac{Q_{s+n} + \alpha^2 + \beta^2}{2W}\right) I_0\left(\frac{\sqrt{Q_{s+n}}}{W} \sqrt{\alpha^2 + \beta^2}\right) \\
 &= \frac{1}{2W} \exp\left(-\frac{Q_{s+n} + \alpha^2 + \beta^2}{2W}\right) I_0\left(\frac{\sqrt{Q_{s+n}}}{W} \sqrt{\alpha^2 + \beta^2}\right) \quad (4.53)
 \end{aligned}$$

Figure 4.4 illustrates this probability density function for two separate values of  $z = (\alpha^2 + \beta^2)/2W$ , similar to to plot of the PDF for the case of the linear detector, Figure 4.2.

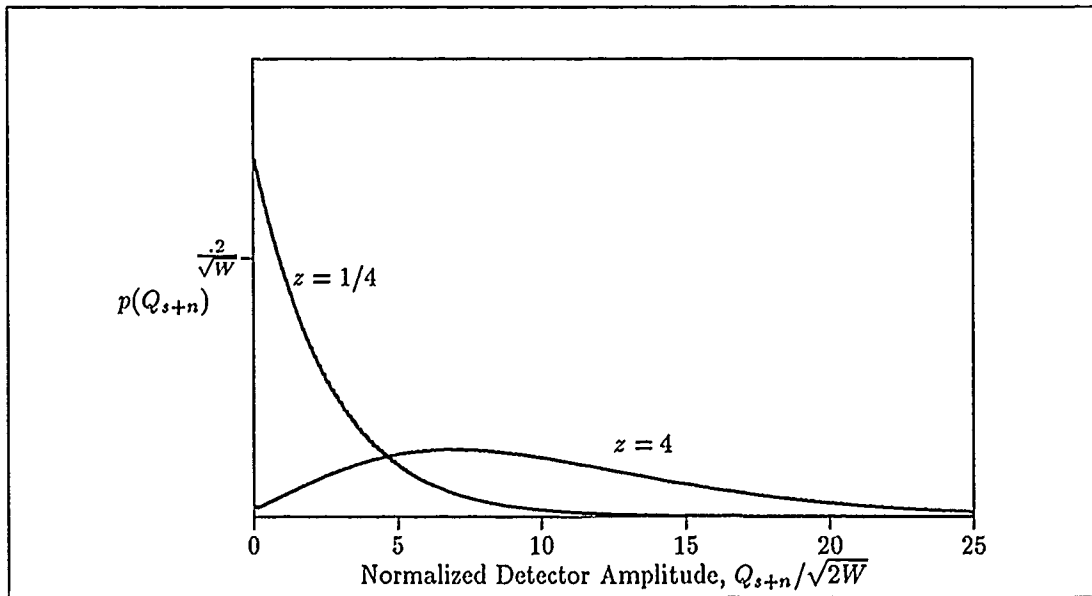


Figure 4.4. PDF of Signal Plus Noise for a Quadratic Detector

**4.2.1 Expected Values** The expected values for the output of a quadratic detector are presented in this section. The expectations of  $Q$  and  $Q^2$  for the cases of noise only and signal plus noise are derived. The autocorrelation function for the signal plus noise case,  $\overline{Q_1 Q_2}$ , is also derived.

The expectations for the noise only case are considered first. With the probability density of  $Q_n$  given by Eq ( 4.52) the average value of the quadratic detector output with only a noise input is

$$\begin{aligned} E(Q_n) &= \int_{-\infty}^{\infty} Q_n p(Q_n) dQ_n \\ &= \int_0^{\infty} \frac{Q_n}{2W} \exp\left(-\frac{Q_n}{2W}\right) dQ_n \end{aligned} \quad (4.54)$$

The integral in the above equation is of the form of Eq (661) from the *CRC Tables* (16:465). Using this equation for the integral, the expectation becomes

$$\begin{aligned} E(Q_n) &= \frac{1}{2W}(2W)^2 \\ &= 2W \end{aligned} \quad (4.55)$$

Next, the mean square value of  $Q_n$ ,  $E(Q_n^2)$ , is calculated from the PDF given by Eq ( 4.52):

$$\begin{aligned} E(Q_n^2) &= \int_{-\infty}^{\infty} Q_n^2 p(Q_n) dQ_n \\ &= \int_0^{\infty} \frac{Q_n^2}{2W} \exp\left(-\frac{Q_n}{2W}\right) dQ_n \end{aligned} \quad (4.56)$$

Again using *CRC* Eq (661), the mean square value is now written

$$\begin{aligned} E(Q_n^2) &= \frac{1}{W}(2W)^3 \\ &= 8W^2 \end{aligned} \quad (4.57)$$

Now the case for signal plus noise is considered. The PDF of the amplitude of the output of the quadratic detector when the input is signal plus noise,  $p(Q_{s+n})$ , is given by Eq ( 4.53). This leads to the expected value of  $Q_{s+n}$ :

$$\begin{aligned}
 E(Q_{s+n}) &= \int_{-\infty}^{\infty} Q_{s+n} p(Q_{s+n}) dQ_{s+n} \\
 &= \int_0^{\infty} \frac{Q_{s+n}}{2W} \exp\left(-\frac{Q_{s+n} + \alpha^2 + \beta^2}{2W}\right) I_0\left(\frac{\sqrt{Q_{s+n}}}{W} \sqrt{\alpha^2 + \beta^2}\right) dQ_{s+n} \\
 &= \frac{1}{W} \exp\left(-\frac{\alpha^2 + \beta^2}{2W}\right) \int_0^{\infty} t^3 \exp\left(-\frac{t^2}{2W}\right) I_0\left(\frac{t}{W} \sqrt{\alpha^2 + \beta^2}\right) dt \quad (4.58)
 \end{aligned}$$

where

$$\begin{aligned}
 t &= \sqrt{Q_{s+n}} \\
 dt &= \frac{1}{2\sqrt{Q_{s+n}}} dQ_{s+n}
 \end{aligned}$$

From Eq ( 4.31) of Section 4.1.3 the expected value becomes

$$\begin{aligned}
 E(Q_{s+n}) &= \frac{1}{W} \exp\left(-\frac{\alpha^2 + \beta^2}{2W}\right) \left[ \frac{\Gamma(2)(2W)^2}{2} \exp\left(\frac{\alpha^2 + \beta^2}{2W}\right) {}_1F_1\left(-1, 1; -\frac{\alpha^2 + \beta^2}{2W}\right) \right] \\
 &= 2W {}_1F_1\left(-1, 1; -\frac{\alpha^2 + \beta^2}{2W}\right) \quad (4.59)
 \end{aligned}$$

where  $\Gamma(2) = 1$  since  $\Gamma(n) = (n-1)!$ . From Eq ( 4.40) in Section 4.1.3, the hypergeometric function in the above equation reduces to the form  $(1-z)$ . Therefore, the expected value of  $Q_{s+n}$  is

$$E(Q_{s+n}) = 2W \left(1 + \frac{\alpha^2 + \beta^2}{2W}\right)$$

$$E(Q_{s+n}) = 2W + \alpha^2 + \beta^2 \quad (4.60)$$

Now, the expected value of  $Q_{s+n}^2$  is derived here. From Eq (4.53), the mean square value is

$$\begin{aligned} E(Q_{s+n}^2) &= \int_{-\infty}^{\infty} Q_{s+n}^2 p(Q_{s+n}) dQ_{s+n} \\ &= \int_0^{\infty} \frac{Q_{s+n}^2}{2W} \exp\left(-\frac{Q_{s+n} + \alpha^2 + \beta^2}{2W}\right) I_0\left(\frac{\sqrt{Q_{s+n}}}{W} \sqrt{\alpha^2 + \beta^2}\right) dQ_{s+n} \\ &= \frac{1}{W} \exp\left(-\frac{\alpha^2 + \beta^2}{2W}\right) \int_0^{\infty} t^5 \exp\left(-\frac{t^2}{2W}\right) I_0\left(\frac{t}{W} \sqrt{\alpha^2 + \beta^2}\right) dt \end{aligned} \quad (4.61)$$

where

$$\begin{aligned} t &= \sqrt{Q_{s+n}} \\ dt &= \frac{1}{2\sqrt{Q_{s+n}}} dQ_{s+n} \end{aligned}$$

As before, from Eq (4.31) the expected value of  $Q_{s+n}^2$  is

$$\begin{aligned} E(Q_{s+n}^2) &= \frac{1}{W} \exp\left(-\frac{\alpha^2 + \beta^2}{2W}\right) \left[ \frac{\Gamma(3)(2W)^3}{2} \exp\left(\frac{\alpha^2 + \beta^2}{2W}\right) {}_1F_1\left(-2, 1; -\frac{\alpha^2 + \beta^2}{2W}\right) \right] \\ &= 3W^2 {}_1F_1\left(-2, 1; -\frac{\alpha^2 + \beta^2}{2W}\right) \end{aligned} \quad (4.62)$$

The confluent hypergeometric series in the above equation can be expanded to a finite series using Eq (4.32) from Section 4.1.3. This expansion is as follows:

$${}_1F_1(-2, 1; z) = 1 - 2z + \frac{z^2}{2} \quad (4.63)$$

This expansion leads to the following form of the mean square value of  $Q_{s+n}$ :

$$\begin{aligned}
E(Q_{s+n}^2) &= 8W^2 \left[ 1 + \frac{\alpha^2 + \beta^2}{W} + \frac{(\alpha^2 + \beta^2)^2}{8W^2} \right] \\
&= 8W^2 + 8W(\alpha^2 + \beta^2) + (\alpha^2 + \beta^2)^2
\end{aligned} \tag{4.64}$$

The last expectation to derive is the autocorrelation function of the output of a quadratic detector. The autocorrelation function is given by

$$\overline{Q_1 Q_2} = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (x_1^2 + y_1^2) (x_2^2 + y_2^2) p(x_1, y_1; x_2, y_2) dx_1 dy_1 dx_2 dy_2 \tag{4.65}$$

where  $p(x_1, y_1; x_2, y_2)$  is given by Eq ( 4.46). Lawson and Uhlenbeck provide the solution to these integrals (11:155). The autocorrelation function of the output of a quadratic detector resolves to

$$\begin{aligned}
\overline{Q_1 Q_2} &= (\alpha_1^2 + \beta_1^2 + 2W) (\alpha_2^2 + \beta_2^2 + 2W) \\
&\quad + 4W\rho(\alpha_1\alpha_2 + \beta_1\beta_2) + 4W^2\rho^2
\end{aligned} \tag{4.66}$$



## V. Effects of CW Interference

In this chapter the effects of single frequency or continuous wave (CW) interference are explored. First, the effects of CW interference on a linear detector is analyzed. Finally, the effects on a quadratic detector are determined. This analysis closely follows the analysis presented by Lawson and Uhlenbeck (11:347-353). For this analysis, a square radar pulse is assumed with pulse width  $\tau$  and amplitude  $S$ . The pulse is defined by the expression

$$s(t) = \begin{cases} S \sin(2\pi f_s t + \theta_s) & , \quad |t| < \frac{\tau}{2} \\ 0 & , \quad \text{elsewhere} \end{cases} \quad (5.1)$$

where

$f_s$  = signal frequency

$\theta_s$  = random signal phase

It is further assumed that the signal frequency,  $f_s$ , is centered in the IF filter passband.

The CW interference is represented by  $E(t)$ :

$$E(t) = E \sin(2\pi f_{cw} t + \theta_{cw}) \quad (5.2)$$

where

$f_{cw}$  = CW interference frequency

$\theta_{cw}$  = random phase of the CW interference

It is assumed that the voltage of the CW interference,  $E$ , is much greater than both the signal voltage,  $S$ , and the average value of the noise voltage. The input to the IF filter (neglecting thermal noise) is the sum of the pulsed signal and the CW interference. Let  $s_{in}(t)$  be the IF filter input:

$$s_{in}(t) = \begin{cases} S \sin(2\pi f_s t + \theta_s) + E \sin(2\pi f_{cw} t + \theta_{cw}) & , \quad |t| < \frac{\tau}{2} \\ E \sin(2\pi f_{cw} t + \theta_{cw}) & , \quad \text{elsewhere} \end{cases} \quad (5.3)$$

It will be convenient to put  $s_{in}(t)$  into quadrature form in terms of the center frequency,  $f_s$ , so that

$$s_{in}(t) = \alpha_o(t) \cos 2\pi f_s t + \beta_o(t) \sin 2\pi f_s t \quad (5.4)$$

For  $|t| < \frac{\tau}{2}$ ,  $s_{in}(t)$  can be manipulated using trigonometric identities into the quadrature form:

$$\begin{aligned} s_{in}(t) &= S(\sin \omega_s t \cos \theta_s + \cos \omega_s t \sin \theta_s) \\ &\quad + E[\sin(\omega' + \omega_s)t \cos \theta_{cw} + \cos(\omega' + \omega_s)t \sin \theta_{cw}] \end{aligned} \quad (5.5)$$

where

$$\omega_s = 2\pi f_s$$

$$\omega' = \omega_{cw} - \omega_s$$

$$\begin{aligned} s_{in}(t) &= S(\sin \omega_s t \cos \theta_s + \cos \omega_s t \sin \theta_s) \\ &\quad + E[(\sin \omega' t \cos \omega_s t + \cos \omega' t \sin \omega_s t) \cos \theta_{cw} \\ &\quad + (\cos \omega' t \cos \omega_s t - \sin \omega' t \sin \omega_s t) \sin \theta_{cw}] \\ &= \cos \omega_s t [S \sin \theta_s + E(\sin \omega' t \cos \theta_{cw} + \cos \omega' t \sin \theta_{cw})] \\ &\quad + \sin \omega_s t [S \cos \theta_s + E(\cos \omega' t \cos \theta_{cw} - \sin \omega' t \sin \theta_{cw})] \\ &= \cos \omega_s t [S \sin \theta_s + E \sin(\omega' t + \theta_{cw})] \\ &\quad + \sin \omega_s t [S \cos \theta_s + E \cos(\omega' t + \theta_{cw})] \end{aligned} \quad (5.6)$$

Likewise, when  $|t| > \frac{\tau}{2}$  the input signal becomes

$$s_{in}(t) = \cos \omega_s t E \sin(\omega' t + \theta_{cw}) + \sin \omega_s t E \cos(\omega' t + \theta_{cw}) \quad (5.7)$$

From Eqs ( 5.6) and ( 5.7)  $\alpha_o(t)$  and  $\beta_o(t)$  can be expressed as

$$\alpha_o(t) = \begin{cases} E \sin [2\pi (f_{cw} - f_s) t + \theta_{cw}] + S \sin \theta_s, & |t| < \frac{\tau}{2} \\ E \sin [2\pi (f_{cw} - f_s) t + \theta_{cw}] & , \text{ elsewhere} \end{cases} \quad (5.8)$$

$$\beta_o(t) = \begin{cases} E \cos [2\pi (f_{cw} - f_s) t + \theta_{cw}] + S \cos \theta_s, & |t| < \frac{\tau}{2} \\ E \cos [2\pi (f_{cw} - f_s) t + \theta_{cw}] & , \text{ elsewhere} \end{cases} \quad (5.9)$$

The analysis by Lawson and Uhlenbeck assumes that the IF filter has a Gaussian passband.

The transfer function of this filter is expressed as

$$H(f) = \exp \left( -\frac{a^2 f^2}{B^2} \right) \quad (5.10)$$

where  $a = 1.18$ , so that the half power bandwidth of the filter is  $B$ . Here, the frequency will be taken about the frequency  $f = f_s$ , which is the center frequency of the IF filter passband. The transfer function of this filter is illustrated in Figure 5.1 for positive frequencies.

The output of the IF filter,  $s_{out}(t)$  can be calculated by Fourier transform techniques as follows

$$s_{out}(t) = \int_{-\infty}^{\infty} S_{in}(f) H(f) e^{-j2\pi f t} df \quad (5.11)$$

where  $S_{in}(f)$  is the Fourier transform of the input to the IF filter.  $S_{in}(f)$  can be manipulated through properties of the Fourier transform to be

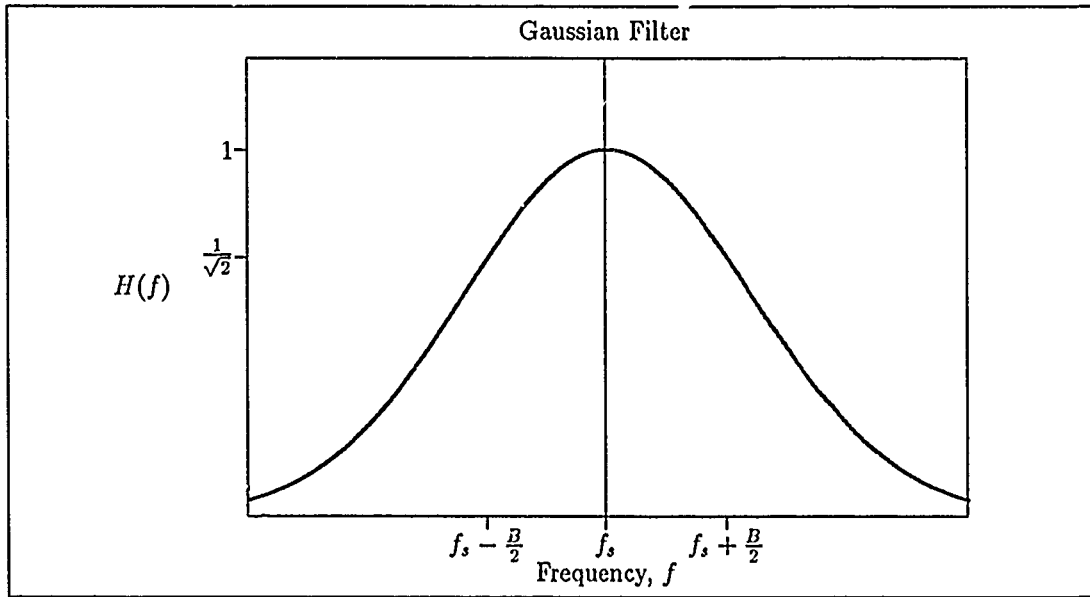


Figure 5.1. IF Filter Transfer Function

$$S_{in}(f) = \frac{1}{2} [A(f - f_s) + A(f + f_s)] + \frac{1}{2j} [B(f - f_s) - B(f + f_s)] \quad (5.12)$$

where

$$A(f) = \int_{-\infty}^{\infty} \alpha_o(t) e^{-j2\pi f t} dt \quad (5.13)$$

$$B(f) = \int_{-\infty}^{\infty} \beta_o(t) e^{-j2\pi f t} dt \quad (5.14)$$

are the Fourier transforms of  $\alpha_o$  and  $\beta_o$  respectively. Using Eq ( 5.12),  $s_{out}(t)$  can be expressed by the following equation:

$$\begin{aligned}
s_{out}(t) &= \frac{1}{2} \int_{-\infty}^{\infty} H(f) [A(f - f_s) + A(f + f_s)] e^{j2\pi f t} df \\
&\quad + \frac{1}{2j} \int_{-\infty}^{\infty} H(f) [B(f - f_s) - B(f + f_s)] e^{j2\pi f t} df
\end{aligned} \tag{5.15}$$

This expression can be manipulated to the form

$$s_{out}(t) = \cos 2\pi f_s t \int_{-\infty}^{\infty} A(f) H(f) e^{j2\pi f t} df + \sin 2\pi f_s t \int_{-\infty}^{\infty} B(f) H(f) e^{j2\pi f t} df \tag{5.16}$$

$$= \alpha(t) \cos 2\pi f_s t + \beta(t) \sin 2\pi f_s t \tag{5.17}$$

where

$$\alpha(t) = \int_{-\infty}^{\infty} A(f) H(f) e^{j2\pi f t} df \tag{5.18}$$

$$\beta(t) = \int_{-\infty}^{\infty} B(f) H(f) e^{j2\pi f t} df \tag{5.19}$$

By substituting Eqs ( 5.8) and ( 5.9) into the above equations  $\alpha(t)$  and  $\beta(t)$  become (11:348)

$$\alpha(t) = \begin{cases} E \exp \left[ -\frac{a^2(f_{cw} - f_s)^2}{B^2} \right] \sin [2\pi (f_{cw} - f_s) t + \theta_{cw}] + S \sin \theta_s, & |t| < \frac{\tau}{2} \\ E \exp \left[ -\frac{a^2(f_{cw} - f_s)^2}{B^2} \right] \sin [2\pi (f_{cw} - f_s) t + \theta_{cw}] & , \text{ elsewhere} \end{cases} \tag{5.20}$$

$$\beta(t) = \begin{cases} E \exp \left[ -\frac{a^2(f_{cw} - f_s)^2}{B^2} \right] \cos [2\pi (f_{cw} - f_s) t + \theta_{cw}] + S \cos \theta_s, & |t| < \frac{\tau}{2} \\ E \exp \left[ -\frac{a^2(f_{cw} - f_s)^2}{B^2} \right] \cos [2\pi (f_{cw} - f_s) t + \theta_{cw}] & , \text{ elsewhere} \end{cases} \tag{5.21}$$

### 5.1 Effects of CW Interference on Linear Detection

In this section, the effects of single frequency CW interference on linear detection are analyzed. Since the CW voltage was assumed to be greater than either the signal or mean noise voltage, the average output of the linear detector,  $\bar{L}$ , when the input is  $s_{out}(t)$ , is found from the approximation of Eq ( 4.37) in Section 4.1.3 to be

$$\bar{L} = \sqrt{\alpha^2 + \beta^2} \quad (5.22)$$

Using the expressions for  $\alpha(t)$  and  $\beta(t)$ , Eqs ( 5.20) and ( 5.21), the average value for  $|t| < \frac{\tau}{2}$  becomes

$$\begin{aligned} \bar{L} &= [X^2 \sin^2(\omega' t + \theta_{cw}) + 2XS \sin(\omega' t + \theta_{cw}) \sin \theta_s + S^2 \sin^2 \theta_s \\ &\quad + X^2 \cos^2(\omega' t + \theta_{cw}) + 2XS \cos(\omega' t + \theta_{cw}) \cos \theta_s + S^2 \cos^2 \theta_s]^{\frac{1}{2}} \\ &= \{X^2 + 2XS [\sin(\omega' t + \theta_{cw}) \sin \theta_s \cos(\omega' t + \theta_{cw}) \cos \theta_s] + S^2\}^{\frac{1}{2}} \end{aligned} \quad (5.23)$$

where

$$\begin{aligned} \omega' &= 2\pi(f_{cw} - f_s) \\ X &= E \exp \left[ -\frac{a^2(f_{cw} - f_s)^2}{B^2} \right] \end{aligned}$$

The above expression for  $\bar{L}$  can be reduced through trigonometric identities to

$$\bar{L} = [X^2 + 2XS \cos(\omega' t + \varphi) + S^2]^{\frac{1}{2}} \quad (5.24)$$

where

$$\varphi = \theta_{cw} - \theta_s$$

It was assumed that the amplitude of the CW interference,  $E$ , was much greater than the amplitude of the signal,  $S$ . Therefore, the  $S^2$  term in the above equation is negligible and  $\bar{L}$  can just as easily be written

$$\begin{aligned}\bar{L} &= [X^2 + 2XS \cos(\omega't + \varphi) + S^2 \cos^2(\omega't + \varphi)]^{\frac{1}{2}} \\ &= X + S \cos(\omega't + \varphi)\end{aligned}\quad (5.25)$$

When  $|t| > \frac{\tau}{2}$ ,  $S = 0$  and  $\bar{L} = X$ . This leads to the following expression for  $\bar{L}$ :

$$\bar{L} = \begin{cases} E \exp \left[ -\frac{a^2(f_{cw} - f_s)^2}{B^2} \right] + S \cos [2\pi (f_{cw} - f_s)t + \varphi] & , \quad |t| < \frac{\tau}{2} \\ E \exp \left[ -\frac{a^2(f_{cw} - f_s)^2}{B^2} \right] & , \quad \text{elsewhere} \end{cases} \quad (5.26)$$

Lawson and Uhlenbeck assume that the video bandwidth,  $b$ , is very narrow, so that only the variation inside of the pulse width,  $\tau$ , needs to be considered (11:349). Therefore,  $\bar{L}$  is time averaged over  $\tau$  to yield  $\tilde{\bar{L}}$ . For the case of the signal present,  $S \neq 0$

$$\begin{aligned}\tilde{\bar{L}} &= \frac{1}{\tau} \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} X + S \cos(\omega't + \varphi) dt \\ &= X + \frac{S}{\omega'\tau} \left[ \sin \left( \frac{\omega'\tau}{2} + \varphi \right) - \sin \left( -\frac{\omega'\tau}{2} + \varphi \right) \right] \\ &= X + \frac{S}{\omega'\tau} \left[ \sin \left( \frac{\omega'\tau}{2} + \varphi \right) + \sin \left( \frac{\omega'\tau}{2} - \varphi \right) \right]\end{aligned}\quad (5.27)$$

where

$$\begin{aligned}\omega' &= 2\pi(f_{cw} - f_s) \\ X &= E \exp \left[ -\frac{a^2(f_{cw} - f_s)^2}{B^2} \right]\end{aligned}$$

Without the signal present,  $S = 0$  and the above equation reduces to  $\tilde{L} = X$ . With this, the equation for  $\tilde{L}$  can be written

$$\tilde{L} = \begin{cases} \tilde{L}_{cw+s} = E \exp \left[ -\frac{a^2(f_{cw}-f_s)^2}{B^2} \right] + \frac{S}{2\pi(f_{cw}-f_s)\tau} \\ \quad \times \{ \sin [\pi(f_{cw}-f_s)\tau + \varphi] + \sin [\pi(f_{cw}-f_s)\tau - \varphi] \} & , \text{ with signal} \\ \tilde{L}_{cw} = E \exp \left[ -\frac{a^2(f_{cw}-f_s)^2}{B^2} \right] & , \text{ without signal} \end{cases} \quad (5.28)$$

To determine the effects of the CW interference, the deflection criterion presented by Lawson and Uhlenbeck is applied. This criterion states that a detection has occurred if the average value of the deflection for the signal plus noise over the average value of the deflection for noise only is on the order of magnitude of the standard deviation of the deflection for noise only (11:161). This criterion in equation form is (11:163)

$$\frac{\bar{y}_{S+N} - \bar{y}_N}{\sqrt{\bar{y}_N^2 - (\bar{y}_N)^2}} = k \quad (5.29)$$

where  $k \approx 1$ . For the case of the linear detector,  $\bar{L}_{S+N}$  is derived by assuming several observations are averaged, and that the phase,  $\varphi$ , is uniformly distributed over the interval  $(0, 2\pi)$ . Further, a central limit theorem argument allows the use of a Gaussian density for a large number of observations of the deflection. The mean of this Gaussian density is  $\tilde{L}_{cw+s} - \tilde{L}_{cw}$  from Eq ( 5.28) and the variance is  $2W$ . Since  $\tilde{L}_{cw+s} - \tilde{L}_{cw}$  will vary between positive and negative values due to phase fluctuations, the average will be taken over one-half of the Gaussian density. As Lawson and Uhlenbeck point out,  $\bar{L}_{S+N}$  is defined by (11:349)

$$\bar{L}_{S+N} = \frac{2}{\sqrt{2\pi W}} \int_0^{2\pi} \frac{1}{2\pi} \int_0^\infty x \exp \left\{ -\frac{[x - (\tilde{L}_{cw+s} - \tilde{L}_{cw})]^2}{2W} \right\} dx d\varphi \quad (5.30)$$



The derivation of  $\bar{L}_{S+N}$  follows. The  $\tilde{\bar{L}}_{cw+s} - \tilde{\bar{L}}_{cw}$  term can be reduced through trigonometric identities to

$$\begin{aligned}\tilde{\bar{L}}_{cw+s} - \tilde{\bar{L}}_{cw} &= \frac{S}{2\pi(f_{cw} - f_s)\tau} \{ \sin [\pi (f_{cw} - f_s) \tau + \varphi] + \sin [\pi (f_{cw} - f_s) \tau - \varphi] \} \\ &= \frac{S}{\pi(f_{cw} - f_s)\tau} \sin [\pi (f_{cw} - f_s) \tau] \cos \varphi \\ &= \gamma \cos \varphi\end{aligned}\quad (5.31)$$

where

$$\gamma = \frac{S}{\pi(f_{cw} - f_s)\tau} \sin [\pi (f_{cw} - f_s) \tau]$$

Substituting the above expression into Eq ( 5.30), gives

$$\begin{aligned}\bar{L}_{S+N} &= \frac{1}{\pi\sqrt{2\pi W}} \int_0^{2\pi} \int_0^\infty x \exp \left[ -\frac{(x - \gamma \cos \varphi)^2}{2W} \right] dx d\varphi \\ &= \frac{1}{\pi\sqrt{2\pi W}} \int_0^{2\pi} \int_{-\frac{\gamma \cos \varphi}{\sqrt{2W}}}^\infty (u\sqrt{2W} + \gamma \cos \varphi) e^{-u^2} \sqrt{2W} du d\varphi \\ &= \frac{1}{\pi\sqrt{\pi}} \int_0^{2\pi} \left( \int_{-\frac{\gamma \cos \varphi}{\sqrt{2W}}}^\infty \sqrt{2W} u e^{-u^2} du + \int_{-\frac{\gamma \cos \varphi}{\sqrt{2W}}}^\infty \gamma \cos \varphi e^{-u^2} du \right) d\varphi\end{aligned}\quad (5.32)$$

where

$$\begin{aligned}u &= \frac{x - \gamma \cos \varphi}{\sqrt{2W}} \\ du &= \frac{dx}{\sqrt{2W}}\end{aligned}$$

The component integrals of the above equation will be solved here. The first integral reduces to

$$\int_{-\frac{\gamma \cos \varphi}{\sqrt{2W}}}^\infty \sqrt{2W} u e^{-u^2} du = \sqrt{2W} \left( \int_0^\infty u e^{-u^2} du + \int_{-\frac{\gamma \cos \varphi}{\sqrt{2W}}}^0 u e^{-u^2} du \right)$$

$$\begin{aligned}
\int_{-\frac{\gamma \cos \varphi}{\sqrt{2W}}}^{\infty} \sqrt{2W} u e^{-u^2} du &= \sqrt{2W} \left\{ \frac{1}{2} + \left[ -\frac{1}{2} + \frac{1}{2} \exp \left( -\frac{\gamma^2 \cos^2 \varphi}{2W} \right) \right] \right\} \\
&= \sqrt{\frac{W}{2}} \exp \left( -\frac{\gamma^2 \cos^2 \varphi}{2W} \right)
\end{aligned} \tag{5.33}$$

The second integral is nearly of the form of the co-error function, and can be written in terms of the co-error function by

$$\begin{aligned}
\int_{-\frac{\gamma \cos \varphi}{\sqrt{2W}}}^{\infty} \gamma \cos \varphi e^{-u^2} du &= \frac{\gamma \sqrt{\pi}}{2} \cos \varphi \left( \frac{2}{\sqrt{\pi}} \int_{-z}^{\infty} e^{-u^2} du \right) \\
&= \frac{\gamma \sqrt{\pi}}{2} \cos \varphi \operatorname{erfc}(-z)
\end{aligned} \tag{5.34}$$

where

$$z = \frac{\gamma \cos \varphi}{\sqrt{2W}}$$

This expression can be reduced to a form containing the error function by the use error function relationships found in the *Handbook of Mathematical Functions* as follows (13:297):

$$\begin{aligned}
\operatorname{erfc}(-z) &= 1 - \operatorname{erf}(-z) \\
\operatorname{erf}(-z) &= -\operatorname{erf}(z) \\
\operatorname{erfc}(-z) &= 1 + \operatorname{erf}(z)
\end{aligned} \tag{5.35}$$

With the above relationship, the integral can be written

$$\int_{-\frac{\gamma \cos \varphi}{\sqrt{2W}}}^{\infty} \gamma \cos \varphi e^{-u^2} du = \frac{\gamma \sqrt{\pi}}{2} \cos \varphi [1 + \operatorname{erf}(z)] \tag{5.36}$$

A series expansion for  $\text{erf}(z)$ , from the *Handbook of Mathematical Functions* is substituted for the error function above (13:297):

$$\begin{aligned} \int_{-\frac{\gamma \cos \varphi}{\sqrt{2W}}}^{\infty} \gamma \cos \varphi e^{-u^2} du &= \frac{\gamma \sqrt{\pi}}{2} \cos \varphi \left[ 1 + \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{n!(2n+1)} \right] \\ &= \frac{\gamma \sqrt{\pi}}{2} \cos \varphi \left[ 1 + \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n \left( \frac{\gamma \cos \varphi}{\sqrt{2W}} \right)^{2n+1}}{n!(2n+1)} \right] \end{aligned} \quad (5.37)$$

Substituting Eqs ( 5.33) and ( 5.37) into Eq ( 5.32),  $\bar{L}_{S+N}$  can be written

$$\begin{aligned} \bar{L}_{S+N} &= \frac{1}{\pi \sqrt{\pi}} \int_0^{2\pi} \left\{ \sqrt{\frac{W}{2}} \exp \left( -\frac{\gamma^2 \cos^2 \varphi}{2W} \right) \right. \\ &\quad \left. + \frac{\gamma \sqrt{\pi}}{2} \cos \varphi \left[ 1 + \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n \left( \frac{\gamma \cos \varphi}{\sqrt{2W}} \right)^{2n+1}}{n!(2n+1)} \right] \right\} d\varphi \\ &= \frac{1}{\pi \sqrt{\pi}} \left\{ \int_0^{2\pi} \sqrt{\frac{W}{2}} \exp \left( -\frac{\gamma^2 \cos^2 \varphi}{2W} \right) d\varphi \right. \\ &\quad \left. + \int_0^{2\pi} \frac{\gamma \sqrt{\pi}}{2} \cos \varphi d\varphi + \int_0^{2\pi} \gamma \cos \varphi \left[ \frac{\gamma \cos \varphi}{\sqrt{2W}} - \frac{\gamma^3 \cos^3 \varphi}{6(2W)^{\frac{3}{2}}} + \dots \right] d\varphi \right\} \end{aligned} \quad (5.38)$$

The argument of the first integral in the above equation can be expanded in a power series of the form of Eq (1.211,3) of Gradshteyn and Ryzhik (8:22). With this series, the first integral becomes

$$\begin{aligned} \int_0^{2\pi} \sqrt{\frac{W}{2}} \exp \left( -\frac{\gamma^2 \cos^2 \varphi}{2W} \right) d\varphi &= \sqrt{\frac{W}{2}} \int_0^{2\pi} \sum_{k=0}^{\infty} \frac{(-1)^k \left( \frac{\gamma^2 \cos^2 \varphi}{2W} \right)^k}{k!} d\varphi \\ &= \sqrt{\frac{W}{2}} \int_0^{2\pi} \left( 1 - \frac{\gamma^2 \cos^2 \varphi}{2W} + \frac{\gamma^4 \cos^4 \varphi}{8W^2} - \dots \right) d\varphi \end{aligned} \quad (5.39)$$

Keeping only the terms up to  $\gamma^2$ , the integral becomes

$$\int_0^{2\pi} \sqrt{\frac{W}{2}} \exp\left(-\frac{\gamma^2 \cos^2 \varphi}{2W}\right) d\varphi = \sqrt{\frac{W}{2}} \left(2\pi - \frac{\gamma^2 \pi}{2W}\right) \quad (5.40)$$

The second integral in Eq ( 5.38) reduces to zero. By keeping only the terms up to  $\gamma^2$ , the third integral becomes

$$\begin{aligned} \int_0^{2\pi} \gamma \cos \varphi \left[ \frac{\gamma \cos \varphi}{\sqrt{2W}} - \frac{\gamma^3 \cos^3 \varphi}{6(2W)^{\frac{3}{2}}} + \dots \right] d\varphi &= \int_0^{2\pi} \frac{\gamma^2 \cos^2 \varphi}{\sqrt{2W}} d\varphi \\ &= \frac{\gamma^2 \pi}{\sqrt{2W}} \end{aligned} \quad (5.41)$$

By substituting the integral results of Eqs ( 5.40) and ( 5.41) into the equation for  $\bar{L}_{S+N}$ , Eq ( 5.38),  $\bar{L}_{S+N}$  becomes

$$\begin{aligned} \bar{L}_{S+N} &= \frac{1}{\pi\sqrt{\pi}} \left[ \sqrt{\frac{W}{2}} \left(2\pi - \frac{\gamma^2 \pi}{2W}\right) + \frac{\gamma^2 \pi}{\sqrt{2W}} \right] \\ &= \sqrt{\frac{2W}{\pi}} \left(1 + \frac{\gamma^2}{4W}\right) \\ &= \sqrt{\frac{2W}{\pi}} \left\{ 1 + \frac{S^2}{4W} \left[ \frac{\sin \pi(f_{cw} - f_s)\tau}{\pi(f_{cw} - f_s)\tau} \right]^2 \right\} \end{aligned} \quad (5.42)$$

The term  $\bar{L}_N$  is derived by setting  $S = 0$  in the above equation, so that

$$\bar{L}_N = \sqrt{\frac{2W}{\pi}} \quad (5.43)$$

Now,  $\bar{L}_{S+N} - \bar{L}_N$  is obtained:

$$\bar{L}_{S+N} - \bar{L}_N = \frac{S^2}{4W} \sqrt{\frac{2W}{\pi}} \left[ \frac{\sin \pi(f_{cw} - f_s)\tau}{\pi(f_{cw} - f_s)\tau} \right]^2 \quad (5.44)$$

Because a very narrow video bandwidth,  $b$ , was assumed, the above equation is reduced by a factor of  $b\tau$  to arrive at the video output:

$$\overline{L_{S+N}} - \overline{L_N} = \frac{S^2 b \tau}{4W} \sqrt{\frac{2W}{\pi}} \left[ \frac{\sin \pi(f_{cw} - f_s)\tau}{\pi(f_{cw} - f_s)\tau} \right]^2 \quad (5.45)$$

To arrive at the standard deviation of the CW and noise only, the power spectral density will first be obtained. The PSD of the CW plus noise is derived by taking the Fourier cosine transform of the autocorrelation of the output of the linear detector. Since it was assumed at the outset that the CW power was much greater than the noise power, the large signal approximation from Section 4.1.3, Eq ( 4.49), is used for the autocorrelation function. As Lawson and Uhlenbeck point out, "since  $E \gg W$ , the main contribution to the continuous spectrum of the linear detector output come from the term proportional to  $\rho$ " (11:350). So, the autocorrelation becomes

$$\overline{L_1 L_2} = W \rho(\tau) \frac{\alpha_1 \alpha_2 + \beta_1 \beta_2}{[(\alpha_1^2 + \beta_1^2)(\alpha_2^2 + \beta_2^2)]^{\frac{1}{2}}} \quad (5.46)$$

where  $\alpha_1$  and  $\alpha_2$  are from Eq ( 5.20) with  $S = 0$  at times  $t_1$  and  $t_2$  respectively. Likewise, the  $\beta$  terms are from Eq ( 5.21) with  $S = 0$ . Now, the autocorrelation becomes

$$\begin{aligned} \overline{L_1 L_2} &= W \rho(\tau) \frac{X^2 \sin \Theta_1 \sin \Theta_2 + X^2 \cos \Theta_1 \cos \Theta_2}{[(X^2 \sin^2 \Theta_1 + X^2 \cos^2 \Theta_1)(X^2 \sin^2 \Theta_2 + X^2 \cos^2 \Theta_2)]^{\frac{1}{2}}} \\ &= W \rho(\tau) (\sin \Theta_1 \sin \Theta_2 + \cos \Theta_1 \cos \Theta_2) \\ &= W \rho(\tau) \cos(\Theta_1 - \Theta_2) \end{aligned} \quad (5.47)$$

where

$$X = E \exp \left[ -\frac{a^2(f_{cw} - f_s)^2}{B^2} \right]$$

$$\Theta_1 = 2\pi(f_{cw} - f_s)t_1 + \theta_{cw}$$

$$\Theta_2 = 2\pi(f_{cw} - f_s)t_2 + \theta_{cw}$$

With the above relationships, the autocorrelation becomes

$$\begin{aligned} \overline{L_1 L_2} &= W \rho(\tau) \cos [2\pi(f_{cw} - f_s)t_1 - 2\pi(f_{cw} - f_s)t_2] \\ &= W \rho(\tau) \cos [2\pi(f_{cw} - f_s)\tau] \end{aligned} \quad (5.48)$$

Next, the normalized correlation coefficient,  $\rho(\tau)$ , is calculated by the use of Eq ( 4.45).

$$\rho(\tau) = \frac{\int_{-\infty}^{\infty} |H(f)|^2 \cos 2\pi f \tau df}{\int_{-\infty}^{\infty} |H(f)|^2 df} \quad (5.49)$$

The IF filter was assumed to have a Gaussian passband whose transfer function,  $H(f)$ , was defined by Eq ( 5.10). With this equation, the numerator in the equation for  $\rho(\tau)$  is

$$\int_{-\infty}^{\infty} |H(f)|^2 \cos 2\pi f \tau df = 2 \int_0^{\infty} \exp \left( -\frac{2a^2 f^2}{B^2} \right) \cos 2\pi f \tau df \quad (5.50)$$

This integral is of the form of Eq (3.896,4) from Gradshteyn and Ryzhik, and resolves to (8:480)

$$\begin{aligned} \int_{-\infty}^{\infty} |H(f)|^2 \cos 2\pi f \tau df &= 2 \left\{ \frac{1}{2} \sqrt{\frac{\pi B^2}{2a^2}} \exp \left[ -\frac{(2\pi\tau)^2 B^2}{4(2a^2)} \right] \right\} \\ &= \frac{B}{a} \sqrt{\frac{\pi}{2}} \exp \left[ -\frac{1}{2} \left( \frac{\pi\tau B}{a} \right)^2 \right] \end{aligned} \quad (5.51)$$

The denominator reduces to

$$\begin{aligned}
 \int_{-\infty}^{\infty} |H(f)|^2 df &= 2 \int_0^{\infty} \exp\left(-\frac{2a^2 f^2}{B^2}\right) df \\
 &= 2 \left(\frac{\sqrt{\pi}}{2}\right) \left(\frac{B}{a\sqrt{2}}\right) \\
 &= \frac{B}{a} \sqrt{\frac{\pi}{2}}
 \end{aligned} \tag{5.52}$$

Combining Eqs ( 5.51) and ( 5.52), the normalized correlation coefficient becomes

$$\rho(\tau) = \exp \left[ -\frac{1}{2} \left( \frac{\pi \tau B}{a} \right)^2 \right] \tag{5.53}$$

With the above the autocorrelation function becomes

$$\overline{L_1 L_2} = R_L(\tau) = W \exp \left[ -\frac{1}{2} \left( \frac{\pi \tau B}{a} \right)^2 \right] \cos \omega' \tau \tag{5.54}$$

where

$$\omega' = 2\pi (f_{cw} - f_s)$$

The power spectral density is calculated by taking the Fourier cosine transform of the autocorrelation function:

$$\begin{aligned}
 S_L(f) &= 2 \int_0^{\infty} R_L(\tau) \cos \omega \tau d\tau \\
 &= 2 \int_0^{\infty} W \exp \left[ -\frac{1}{2} \left( \frac{\pi \tau B}{a} \right)^2 \right] \cos \omega' \tau \cos \omega \tau d\tau
 \end{aligned}$$

$$\begin{aligned}
S_L(f) = & W \int_0^\infty \exp \left[ -\frac{1}{2} \left( \frac{\pi \tau B}{a} \right)^2 \right] \cos(\omega' - \omega) \tau d\tau \\
& + W \int_0^\infty \exp \left[ -\frac{1}{2} \left( \frac{\pi \tau B}{a} \right)^2 \right] \cos(\omega' + \omega) \tau d\tau
\end{aligned} \quad (5.55)$$

The integrals in the above equation are of the form of Eq (3.896,4) found in Gradshteyn and Ryzhik (8:480). Using this equation, the PSD reduces to

$$\begin{aligned}
S_L(f) &= \frac{W}{2} \sqrt{\frac{2a^2\pi}{\pi^2 B^2}} \left\{ \exp \left[ -\frac{a^2(\omega' - \omega)^2}{2\pi^2 B^2} \right] + \exp \left[ -\frac{a^2(\omega' + \omega)^2}{2\pi^2 B^2} \right] \right\} \\
&= \frac{Wa}{2B} \sqrt{\frac{2}{\pi}} \left\{ \exp \left[ -\frac{2a^2(f_{cw} - f_s - f)^2}{B^2} \right] + \exp \left[ -\frac{2a^2(f_{cw} - f_s + f)^2}{B^2} \right] \right\} \quad (5.56)
\end{aligned}$$

This PSD is illustrated in Figure 5.2.

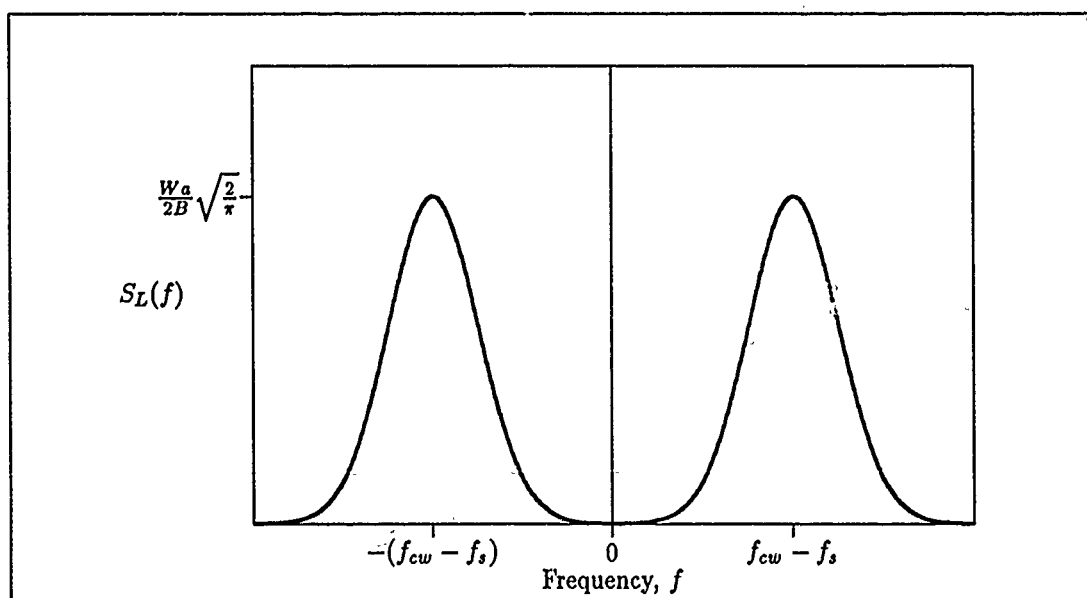


Figure 5.2. PSD of the Linear Detector Output



Since a narrow video bandwidth,  $b$ , was assumed, the variance of the CW plus noise video is  $2S_L(0)b$ . Therefore, from Eq ( 5.56) the variance of the CW plus noise video output is

$$\begin{aligned}\overline{L_N^2} - (\overline{L_N})^2 &= 2b \frac{Wa}{2B} \sqrt{\frac{2}{\pi}} \left\{ \exp \left[ -\frac{2a^2 (f_{cw} - f_s)^2}{B^2} \right] + \exp \left[ -\frac{2a^2 (f_{cw} - f_s)^2}{B^2} \right] \right\} \\ &= \frac{2aWb}{B} \sqrt{\frac{2}{\pi}} \exp \left[ -\frac{2a^2 (f_{cw} - f_s)^2}{B^2} \right]\end{aligned}\quad (5.57)$$

The standard deviation of the video output is simply the square root of the variance. So, the square root of Eq ( 5.57) is

$$\left[ \overline{L_N^2} - (\overline{L_N})^2 \right]^{\frac{1}{2}} = \left( \frac{2}{\pi} \right)^{\frac{1}{4}} \sqrt{\frac{2aWb}{B}} \exp \left[ -\frac{a^2 (f_{cw} - f_s)^2}{B^2} \right] \quad (5.58)$$

From the equation for the deflection criterion, Eq ( 5.29), an expression for the minimum detectable average signal power is derived. The deflection criterion is represented by

$$\frac{\overline{L_{S+N}} - \overline{L_N}}{\left[ \overline{L_N^2} - (\overline{L_N})^2 \right]^{\frac{1}{2}}} = C \quad (5.59)$$

where  $C$  is a constant determined by the radar receiver structure. Substituting Eqs ( 5.45) and ( 5.58) into the above equation yields

$$\frac{S^2 b \tau}{4W} \sqrt{\frac{2W}{\pi}} \left[ \frac{\sin \pi (f_{cw} - f_s) \tau}{\pi (f_{cw} - f_s) \tau} \right]^2 = C \left( \frac{2}{\pi} \right)^{\frac{1}{4}} \sqrt{\frac{2aWb}{B}} \exp \left[ -\frac{a^2 (f_{cw} - f_s)^2}{B^2} \right] \quad (5.60)$$

Lawson and Uhlenbeck define the minimum detectable average signal power as  $\overline{P}_{\min} = \frac{S^2 \tau}{T_o}$ , where  $T_o$  is the pulse recurrence interval (11:350). Now  $\overline{P}_{\min}$  is derived:

$$\begin{aligned}
\bar{P}_{\min} &= \frac{S^2 \tau}{T_o} \\
&= \frac{4CW}{bT_o} \sqrt{\frac{ab\pi}{B}} \left(\frac{2}{\pi}\right)^{\frac{1}{4}} \exp \left[ -\frac{a^2 (f_{cw} - f_s)^2}{B^2} \right] \left[ \frac{\pi (f_{cw} - f_s) \tau}{\sin \pi (f_{cw} - f_s) \tau} \right]^2 \quad (5.61)
\end{aligned}$$

From Eq ( 4.4) of Section 4.1.1,  $W$  can be written in terms of the IF bandwidth,  $B$ , by

$$\begin{aligned}
W &= \sigma^2 \int_{-\infty}^{\infty} |H(f)|^2 df \\
&= 2\sigma^2 \int_0^{\infty} \exp \left( -\frac{2a^2 f^2}{B^2} \right) df \\
&= \frac{\sigma^2 B}{a} \sqrt{\frac{\pi}{2}} \quad (5.62)
\end{aligned}$$

Substituting the above result into Eq ( 5.61) the minimum detectable average signal power becomes

$$\begin{aligned}
\bar{P}_{\min} &= \frac{4\sigma^2 CB}{abT_o} \sqrt{\frac{\pi}{2}} \sqrt{\frac{ab\pi}{B}} \left(\frac{2}{\pi}\right)^{\frac{1}{4}} \exp \left[ -\frac{a^2 (f_{cw} - f_s)^2}{B^2} \right] \left[ \frac{\pi (f_{cw} - f_s) \tau}{\sin \pi (f_{cw} - f_s) \tau} \right]^2 \\
&= \frac{4\sigma^2 C}{T_o} \sqrt{\frac{\pi B}{2ab}} (2\pi)^{\frac{1}{4}} \exp \left[ -\frac{a^2 (f_{cw} - f_s)^2}{B^2} \right] \left[ \frac{\pi (f_{cw} - f_s) \tau}{\sin \pi (f_{cw} - f_s) \tau} \right]^2 \quad (5.63)
\end{aligned}$$

To quantify the effects of on-frequency CW interference, the minimum average power required for the case of large detuning,  $\bar{P}_{\min}(\infty)$ , is first derived. Large detuning implies that  $|f_{cw} - f_s| > B$ , and in this case the CW interference can be considered to be nonexistent (11:351). This derivation of  $\bar{P}_{\min}(\infty)$  mirrors that of the derivation for the minimum signal power when the CW was present. First, the numerator of the expression for the deflection criterion,  $\bar{L}_{S+N} - \bar{L}_S$  is derived.  $\bar{L}_{S+N}$  is obtained from the equation for the expectation of Eq ( 4.34) in Section 4.1.3:

$$\bar{L}_{S+N} = \sqrt{\frac{\pi W}{2}} {}_1F_1 \left( -\frac{1}{2}, 1; -\frac{\alpha^2 + \beta^2}{2W} \right) \quad (5.64)$$

From the equations for  $\alpha(t)$  and  $\beta(t)$ , Eqs ( 5.20) and ( 5.21), without the CW signal, the term  $\alpha^2 + \beta^2 = S^2$ . With this, the expected value is

$$\bar{L}_{S+N} = \sqrt{\frac{\pi W}{2}} {}_1F_1 \left( -\frac{1}{2}, 1; -\frac{S^2}{2W} \right) \quad (5.65)$$

Here, a small signal-to-noise ratio is assumed ( $S^2/2W \ll 1$ ). So, keeping only the first two terms of the expansion for  ${}_1F_1 \left( -\frac{1}{2}, 1; -\frac{S^2}{2W} \right)$ , Eq ( 4.32) from Section 4.1.3,  $\bar{L}_{S+N}$  becomes

$$\bar{L}_{S+N} = \sqrt{\frac{\pi W}{2}} \left( 1 + \frac{S^2}{4W} \right) \quad (5.66)$$

The expected value for the noise only case is given by Eq ( 4.26):

$$\bar{L}_N = \sqrt{\frac{\pi W}{2}} \quad (5.67)$$

Combining the above two equations leads to

$$\begin{aligned} \bar{L}_{S+N} - \bar{L}_N &= \sqrt{\frac{\pi W}{2}} \left( 1 + \frac{S^2}{4W} \right) - \sqrt{\frac{\pi W}{2}} \\ &= \frac{S^2}{4W} \sqrt{\frac{\pi W}{2}} \end{aligned} \quad (5.68)$$

As before, because of a narrow video bandwidth, the above equation is reduced by a factor of  $b\tau$ :

$$\bar{L}_{S+N} - \bar{L}_N = \frac{S^2 b\tau}{4W} \sqrt{\frac{\pi W}{2}} \quad (5.69)$$

Next, the denominator for the deflection criterion,  $\left[\overline{L_N^2} - (\overline{L_N})^2\right]^{\frac{1}{2}}$  is obtained from the PSD for the noise only case. First, the Fourier cosine transform of the autocorrelation function,  $R_L(\tau)$  is taken to obtain the PSD,  $S_L(f)$ . Eq ( 4.48) gives the small signal-to-noise ratio approximation for  $\overline{L_1 L_2}$ . For the noise only case,  $S = 0$  and  $\alpha(t) = \beta(t) = 0$ . Therefore, the only applicable term of this approximation is the term proportional to  $\rho^2$ , and the autocorrelation function becomes

$$\overline{L_1 L_2} = R_L(\tau) = \frac{\pi W \rho^2(\tau)}{8} \quad (5.70)$$

The normalized correlation coefficient,  $\rho(\tau)$ , was defined by Eq ( 5.53). Taking the Fourier cosine transform of the above equation yields the PSD:

$$\begin{aligned} S_L(f) &= 2 \int_0^\infty R_L(\tau) \cos \omega \tau d\tau \\ &= 2 \int_0^\infty \frac{\pi W \rho^2(\tau)}{8} \cos \omega \tau d\tau \\ &= \frac{4\pi}{W} \int_0^\infty \exp \left[ - \left( \frac{\pi \tau B}{a} \right)^2 \right] \cos \omega \tau d\tau \end{aligned} \quad (5.71)$$

The above integral is of the form of Eq (679) from the *CRC Tables* and reduces to (16:466)

$$S_L(f) = \frac{Wab\sqrt{\pi}}{8B} \exp \left( - \frac{a^2 f^2}{B^2} \right) \quad (5.72)$$

As before for a narrow video bandwidth, the variance is  $2S_L(0)b$ :

$$\overline{L_N^2} - (\overline{L_N})^2 = \frac{Wab\sqrt{\pi}}{4B} \quad (5.73)$$

Substituting Eqs ( 5.69) and ( 5.73) into the expression for the deflection criterion one finds

$$\frac{S^2 b \tau}{4W} \sqrt{\frac{\pi W}{2}} = C \left( \frac{W a b \sqrt{\pi}}{4B} \right)^{\frac{1}{2}} \quad (5.74)$$

where  $W$  was defined by Eq ( 5.62). This equation can be reduced to the expression for  $\bar{P}_{\min}(\infty)$  as was done for the case with CW present:

$$\bar{P}_{\min}(\infty) = \frac{4\sigma^2 C}{T_o} \sqrt{\frac{\pi B}{2ab}} (4\pi)^{-\frac{1}{2}} \quad (5.75)$$

For on-frequency CW, ( $f_{cw} = f_s$ ), the required minimum average signal power relative to the power required when no CW is present is given by

$$\begin{aligned} \frac{\bar{P}_{\min}(0)}{\bar{P}_{\min}(\infty)} &= (8\pi^2)^{\frac{1}{2}} \\ &= 4.75 \text{ dB} \end{aligned} \quad (5.76)$$

Lawson and Uhlenbeck compare this value with a value of 3.5 dB obtained experimentally by A. L. Gardener and C. M. Allred (11:351). Lawson and Uhlenbeck also compare experimental data with theoretical calculations for various values of  $\tau|f_{cw} - f_s|$ . The experimental data was collected with a narrow video bandwidth and a large IF bandwidth where  $B\tau = 2.2$  (11:351). This data is displayed in Table 5.1. As Lawson and Uhlenbeck point out, the theoretical values of infinity for integer values of  $\tau|f_{cw} - f_s|$  arise because  $S^2$  terms were neglected (11:352). Further, theoretical values for  $\tau|f_{cw} - f_s| > 2.5$  were not calculated since the equation for  $\bar{P}_{\min}$  is not valid for values of  $|f_{cw} - f_s| > B$  (11:352). The theoretical values in Table 5.1 are calculated from the following equation:

$$\frac{\bar{P}_{\min}}{\bar{P}_{\min}(0)} = \exp \left[ -\frac{a^2 (f_{cw} - f_s)^2}{B^2} \right] \left[ \frac{\pi (f_{cw} - f_s) \tau}{\sin \pi (f_{cw} - f_s) \tau} \right]^2 \quad (5.77)$$

Table 5.1. Theoretical vs Experimental Values of Signal Threshold Due to CW Detuning for a Linear Detector

$\tau f_{cw} - f_s $	$\bar{P}_{\min}/\bar{P}_{\min}(0)$ , dB	
	Experimental	Theoretical
0.5	5.5	3.6
1	13.5	$\infty$
1.5	11	10.75
2	15	$\infty$
2.5	9	10.3
3	5	
3.5	0	
4	-2.5	
4.5	-3.5	

(11:352)

## 5.2 Effects of CW Interference on Quadratic Detection

In this section an analysis of the effects of CW interference on quadratic detection is accomplished. The analysis here closely follows that of the analysis of the effects on linear detection of the previous section. The average output of the quadratic detector for the case of signal plus noise is given by Eq ( 4.60) of Section 4.2.1:

$$\bar{Q} = 2W + \alpha^2 + \beta^2 \quad (5.78)$$

Substituting the expressions for  $\alpha(t)$  and  $\beta(t)$  , Eqs ( 5.20) and ( 5.21), the average value of the detector output for  $|t| < \frac{\tau}{2}$  becomes

$$\begin{aligned}
 \bar{Q} &= 2W + X^2 \sin^2(\omega't + \theta_{cw}) + 2XS \sin(\omega't + \theta_{cw}) \sin \theta_s + S^2 \sin^2 \theta_s \\
 &\quad + X^2 \cos^2(\omega't + \theta_{cw}) + 2XS \cos(\omega't + \theta_{cw}) \cos \theta_s + S^2 \cos^2 \theta_s \\
 &= 2W + X^2 + 2XS [\sin(\omega't + \theta_{cw}) \sin \theta_s \cos(\omega't + \theta_{cw}) \cos \theta_s] + S^2 \quad (5.79)
 \end{aligned}$$

where

$$\omega' = 2\pi(f_{cw} - f_s)$$

$$X = E \exp \left[ -\frac{a^2(f_{cw} - f_s)^2}{B^2} \right]$$

By the use of trigonometric identities, the above expression for  $\overline{Q}$  can be reduced to the following form:

$$\overline{Q} = 2W + X^2 + 2XS \cos(\omega't + \varphi) + S^2 \quad (5.80)$$

where

$$\varphi = \theta_{cw} - \theta_s$$

Since it was assumed that  $E \gg S$ , the  $S^2$  term in the above equation is negligible. Therefore, neglecting the  $S^2$  term,  $\overline{Q}$  can be written

$$\overline{Q} = 2W + X^2 + 2XS \cos(\omega't + \varphi) \quad (5.81)$$

When  $|t| > \frac{\tau}{2}$ ,  $S = 0$  and  $\overline{Q} = 2W + X^2$ . This leads to the following expression for  $\overline{Q}$ :

$$\overline{Q} = \begin{cases} 2W + E^2 \exp \left[ -\frac{2a^2(f_{cw} - f_s)^2}{B^2} \right] \\ \quad + 2ES \exp \left[ -\frac{a^2(f_{cw} - f_s)^2}{B^2} \right] \cos [2\pi(f_{cw} - f_s)t + \varphi] & , \quad |t| < \frac{\tau}{2} \\ 2W + E^2 \exp \left[ -\frac{2a^2(f_{cw} - f_s)^2}{B^2} \right] & , \quad \text{elsewhere} \end{cases} \quad (5.82)$$

For a very narrow video bandwidth,  $b$ , only the variation inside of the pulse width,  $\tau$ , needs to be considered. Therefore,  $\overline{Q}$  is time averaged over the pulse width,  $\tau$ , to arrive at  $\tilde{\overline{Q}}$ . For the case of the signal present,  $S \neq 0$

$$\begin{aligned}
\tilde{\tilde{Q}} &= \frac{1}{\tau} \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} 2W + X^2 + 2XS \cos(\omega't + \varphi) dt \\
&= 2W + X^2 + \frac{2XS}{\omega'\tau} \left[ \sin\left(\frac{\omega'\tau}{2} + \varphi\right) - \sin\left(-\frac{\omega'\tau}{2} + \varphi\right) \right] \\
&= 2W + X^2 + \frac{2XS}{\omega'\tau} \left[ \sin\left(\frac{\omega'\tau}{2} + \varphi\right) + \sin\left(\frac{\omega'\tau}{2} - \varphi\right) \right]
\end{aligned} \tag{5.83}$$

where

$$\begin{aligned}
\omega' &= 2\pi(f_{cw} - f_s) \\
X &= E \exp \left[ -\frac{a^2(f_{cw} - f_s)^2}{B^2} \right]
\end{aligned}$$

When no signal is present,  $S = 0$ , and the above equation reduces to  $\tilde{\tilde{Q}} = 2W + X^2$ . With these expressions, the equation for  $\tilde{\tilde{Q}}$  becomes

$$\tilde{\tilde{Q}} = \begin{cases} \tilde{\tilde{Q}}_{cw+s} = 2W + E^2 \exp \left[ -\frac{2a^2(f_{cw} - f_s)^2}{B^2} \right] + \frac{SE}{\pi(f_{cw} - f_s)\tau} \exp \left[ -\frac{a^2(f_{cw} - f_s)^2}{B^2} \right] \\ \quad \times \{ \sin[\pi(f_{cw} - f_s)\tau + \varphi] + \sin[\pi(f_{cw} - f_s)\tau - \varphi] \} & , \text{ with signal} \\ \tilde{\tilde{Q}}_{cw} = 2W + E^2 \exp \left[ -\frac{2a^2(f_{cw} - f_s)^2}{B^2} \right] & , \text{ without signal} \end{cases} \tag{5.84}$$

As was the case for the analysis of the linear detector, the deflection criterion will be applied.

As before,  $\overline{Q}_{S+N}$  is defined by (11:349):

$$\overline{Q}_{S+N} = \frac{2}{\sqrt{2\pi W}} \int_0^{2\pi} \frac{1}{2\pi} \int_0^\infty x \exp \left\{ -\frac{[x - (\tilde{\tilde{Q}}_{cw+s} - \tilde{\tilde{Q}}_{cw})]^2}{2W} \right\} dx d\varphi \tag{5.85}$$

Now,  $\overline{Q}_{S+N}$  is derived. The  $\tilde{\tilde{Q}}_{cw+s} - \tilde{\tilde{Q}}_{cw}$  term reduces by way of trigonometric identities to



$$\begin{aligned}
\tilde{Q}_{cw+s} - \tilde{Q}_{cw} &= \frac{SE}{\pi(f_{cw} - f_s)\tau} \exp \left[ -\frac{a^2(f_{cw} - f_s)^2}{B^2} \right] \\
&\quad \times \{ \sin [\pi (f_{cw} - f_s) \tau + \varphi] + \sin [\pi (f_{cw} - f_s) \tau - \varphi] \} \\
&= \frac{2SE}{\pi(f_{cw} - f_s)\tau} \exp \left[ -\frac{a^2(f_{cw} - f_s)^2}{B^2} \right] \sin [\pi (f_{cw} - f_s) \tau] \cos \varphi \\
&= \gamma \cos \varphi
\end{aligned} \tag{5.86}$$

where

$$\gamma = \frac{2SE}{\pi(f_{cw} - f_s)\tau} \exp \left[ -\frac{a^2(f_{cw} - f_s)^2}{B^2} \right] \sin [\pi (f_{cw} - f_s) \tau]$$

Substituting the above expression for  $\tilde{Q}_{cw+s} - \tilde{Q}_{cw}$  into Eq ( 5.85), gives the following equation for  $\overline{Q}_{S+N}$ :

$$\overline{Q}_{S+N} = \frac{1}{\pi\sqrt{2\pi W}} \int_0^{2\pi} \int_0^\infty x \exp \left[ -\frac{(x - \gamma \cos \varphi)^2}{2W} \right] dx d\varphi \tag{5.87}$$

The integrals in the above equation were solved in Section 5.1. They reduce in terms of  $\gamma$  to the form of Eq ( 5.42):

$$\overline{Q}_{S+N} = \sqrt{\frac{2W}{\pi}} \left( 1 + \frac{\gamma^2}{4W} \right) \tag{5.88}$$

Substituting the expression for  $\gamma$  into the above equation yields the following equation for  $\overline{Q}_{S+N}$ :

$$\overline{Q}_{S+N} = \sqrt{\frac{2W}{\pi}} \left\{ 1 + \frac{S^2 E^2}{W} \exp \left[ -\frac{2a^2(f_{cw} - f_s)^2}{B^2} \right] \left[ \frac{\sin \pi (f_{cw} - f_s) \tau}{\pi (f_{cw} - f_s) \tau} \right]^2 \right\} \tag{5.89}$$

The term  $\overline{Q}_N$  is derived by setting  $S = 0$  in the above equation, so that

$$\overline{Q}_N = \sqrt{\frac{2W}{\pi}} \quad (5.90)$$

Now,  $\overline{Q}_{S+N} - \overline{Q}_N$  is obtained:

$$\overline{Q}_{S+N} - \overline{Q}_N = \frac{S^2 E^2}{W} \sqrt{\frac{2W}{\pi}} \exp \left[ -\frac{2a^2(f_{cw} - f_s)^2}{B^2} \right] \left[ \frac{\sin \pi(f_{cw} - f_s)\tau}{\pi(f_{cw} - f_s)\tau} \right]^2 \quad (5.91)$$

Since  $b$ , the video bandwidth, was assumed to be very narrow, the above equation is reduced by a factor of  $b\tau$ . Therefore,  $\overline{Q}_{S+N} - \overline{Q}_N$  becomes

$$\overline{Q}_{S+N} - \overline{Q}_N = \frac{S^2 E^2 b\tau}{W} \sqrt{\frac{2W}{\pi}} \exp \left[ -\frac{2a^2(f_{cw} - f_s)^2}{B^2} \right] \left[ \frac{\sin \pi(f_{cw} - f_s)\tau}{\pi(f_{cw} - f_s)\tau} \right]^2 \quad (5.92)$$

To arrive at the standard deviation of the CW and noise only, the power spectral density will first be obtained from the Fourier cosine transform of the autocorrelation function. The autocorrelation function is given by Eq ( 4.66) of Section 4.2.1. Since the only term contributing to the continuous spectrum of the output is that term proportional to  $\rho$ , the autocorrelation becomes

$$\overline{Q_1 Q_2} = 4W\rho(\tau) (\alpha_1 \alpha_2 + \beta_1 \beta_2) \quad (5.93)$$

where  $\alpha_1$  and  $\alpha_2$  are from Eq ( 5.20) with  $S = 0$  at times  $t_1$  and  $t_2$  respectively. Likewise, the  $\beta$  terms are from Eq ( 5.21) with  $S = 0$ . Now, the autocorrelation becomes

$$\begin{aligned} \overline{Q_1 Q_2} &= 4W\rho(\tau) X^2 (\sin \Theta_1 \sin \Theta_2 + \cos \Theta_1 \cos \Theta_2) \\ &= 4W\rho(\tau) X^2 \cos(\Theta_1 - \Theta_2) \end{aligned} \quad (5.94)$$

where

$$X = E \exp \left[ -\frac{a^2(f_{cw} - f_s)^2}{B^2} \right]$$

$$\Theta_1 = 2\pi(f_{cw} - f_s)t_1 + \theta_{cw}$$

$$\Theta_2 = 2\pi(f_{cw} - f_s)t_2 + \theta_{cw}$$

With the above relationships, the autocorrelation becomes

$$\begin{aligned} \overline{Q_1 Q_2} &= 4WE^2 \exp \left[ -\frac{2a^2(f_{cw} - f_s)^2}{B^2} \right] \rho(\tau) \cos [2\pi(f_{cw} - f_s)t_1 - 2\pi(f_{cw} - f_s)t_2] \\ &= 4WE^2 \exp \left[ -\frac{2a^2(f_{cw} - f_s)^2}{B^2} \right] \rho(\tau) \cos [2\pi(f_{cw} - f_s)\tau] \end{aligned} \quad (5.95)$$

The normalized correlation coefficient,  $\rho(\tau)$ , was calculated in Section 5.1 by the use of Eq (4.45):

$$\rho(\tau) = \exp \left[ -\frac{1}{2} \left( \frac{\pi \tau B}{a} \right)^2 \right] \quad (5.96)$$

With the above equation, the autocorrelation function becomes

$$\overline{Q_1 Q_2} = R_Q(\tau) = 4WE^2 \exp \left[ -\frac{2a^2(f_{cw} - f_s)^2}{B^2} \right] \exp \left[ -\frac{1}{2} \left( \frac{\pi \tau B}{a} \right)^2 \right] \cos \omega' \tau \quad (5.97)$$

where

$$\omega' = 2\pi(f_{cw} - f_s)$$

The power spectral density is calculated by taking the Fourier cosine transform of the autocorrelation function:

$$\begin{aligned}
S_Q(f) &= 2 \int_0^\infty R_Q(\tau) \cos \omega \tau d\tau \\
&= 8WE^2 \exp \left[ -\frac{2a^2(f_{cw} - f_s)^2}{B^2} \right] \int_0^\infty \exp \left[ -\frac{1}{2} \left( \frac{\pi \tau B}{a} \right)^2 \right] \cos \omega' \tau \cos \omega \tau d\tau \quad (5.98)
\end{aligned}$$

The integral in the above equation was solved in Section 5.1, Eqs (5.55) and (5.56). Using these relationships, the power spectral density reduces to

$$\begin{aligned}
S_Q(f) &= \frac{2WaE^2}{B} \sqrt{\frac{2}{\pi}} \exp \left[ -\frac{2a^2(f_{cw} - f_s)^2}{B^2} \right] \left\{ \exp \left[ -\frac{2a^2(f_{cw} - f_s - f)^2}{B^2} \right] \right. \\
&\quad \left. + \exp \left[ -\frac{2a^2(f_{cw} - f_s + f)^2}{B^2} \right] \right\} \quad (5.99)
\end{aligned}$$

This PSD is illustrated in Figure 5.3.

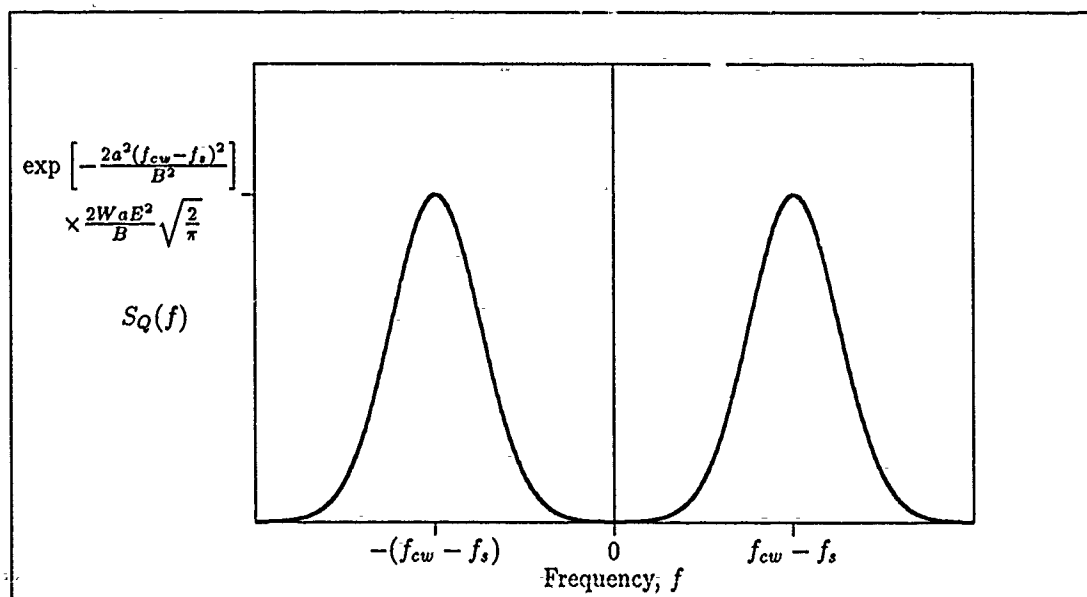


Figure 5.3. PSD of the Output of a Quadratic Detector

Because a very narrow video bandwidth,  $b$ , was assumed, the variance of the CW plus noise video output is  $2S_Q(0)b$ . Therefore, from Eq ( 5.99) the variance of the CW plus noise video output is

$$\overline{Q_N^2} - (\overline{Q_N})^2 = \frac{8aWbE^2}{B} \sqrt{\frac{2}{\pi}} \exp \left[ -\frac{4a^2(f_{cw} - f_s)^2}{B^2} \right] \quad (5.100)$$

The standard deviation of the video output is simply the square root of the variance. So, the square root of Eq ( 5.100) is

$$\left[ \overline{Q_N^2} - (\overline{Q_N})^2 \right]^{\frac{1}{2}} = \left( \frac{2}{\pi} \right)^{\frac{1}{4}} 2E \sqrt{\frac{2aWb}{B}} \exp \left[ -\frac{2a^2(f_{cw} - f_s)^2}{B^2} \right] \quad (5.101)$$

From the equation for the deflection criterion, Eq ( 5.29), an expression for the minimum detectable average signal power is derived. The deflection criterion is represented by

$$\frac{\overline{Q_{s+N}} - \overline{Q_N}}{\left[ \overline{Q_N^2} - (\overline{Q_N})^2 \right]^{\frac{1}{2}}} = C \quad (5.102)$$

where  $C$  is a constant determined by the radar receiver structure. Substituting Eqs ( 5.92) and ( 5.101) into the above equation yields

$$\frac{S^2 E^2 b \tau}{W} \sqrt{\frac{2W}{\pi}} \exp \left[ -\frac{2a^2(f_{cw} - f_s)^2}{B^2} \right] \left[ \frac{\sin \pi(f_{cw} - f_s)\tau}{\pi(f_{cw} - f_s)\tau} \right]^2 = C \left( \frac{2}{\pi} \right)^{\frac{1}{4}} 2E \sqrt{\frac{2aWb}{B}} \exp \left[ -\frac{2a^2(f_{cw} - f_s)^2}{B^2} \right] \quad (5.103)$$

The minimum detectable average signal power was defined as  $\overline{P}_{\min} = \frac{S^2 \tau}{T_o}$ , where  $T_o$  is the pulse recurrence interval (11:350). Now,  $\overline{P}_{\min}$  for the quadratic detector is derived:

$$\begin{aligned}
\bar{P}_{\min} &= \frac{S^2 \tau}{T_o} \\
&= \frac{2CW}{EbT_o} \sqrt{\frac{ab\pi}{B}} \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \left[ \frac{\pi(f_{cw} - f_s)\tau}{\sin \pi(f_{cw} - f_s)\tau} \right]^2
\end{aligned} \tag{5.104}$$

$W$  was derived in terms of the IF bandwidth  $B$  in section 5.1, and is given by Eq ( 5.62) where

$$W = \frac{\sigma^2 B}{a} \sqrt{\frac{\pi}{2}} \tag{5.105}$$

Substituting the above result into Eq ( 5.104) the minimum detectable average signal power becomes

$$\begin{aligned}
\bar{P}_{\min} &= \frac{2\sigma^2 CB}{EabT_o} \sqrt{\frac{\pi}{2}} \sqrt{\frac{ab\pi}{B}} \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \left[ \frac{\pi(f_{cw} - f_s)\tau}{\sin \pi(f_{cw} - f_s)\tau} \right]^2 \\
&= \frac{2\sigma^2 C}{T_o E} \sqrt{\frac{\pi B}{2ab}} (2\pi)^{\frac{1}{2}} \left[ \frac{\pi(f_{cw} - f_s)\tau}{\sin \pi(f_{cw} - f_s)\tau} \right]^2
\end{aligned} \tag{5.106}$$

An analysis of the effects of the CW interference similar to that for the case of a linear detector is accomplished here.  $\bar{P}_{\min}(\infty)$  is derived to quantify the effects of on-frequency jamming. The numerator of the expression for the deflection criterion,  $\bar{Q}_{s+N} - \bar{Q}_N$  is derived first.  $\bar{Q}_{s+N}$  is obtained from Eq ( 4.60) of Section 4.2.1:

$$\bar{Q}_{s+N} = 2W + \alpha^2 + \beta^2 \tag{5.107}$$

From the expressions for  $\alpha(t)$  and  $\beta(t)$ , Eqs ( 5.20) and ( 5.21),  $\alpha^2 + \beta^2 = S^2$  when no CW interference is present. This leads to

$$\overline{Q}_{S+N} = 2W + S^2 \quad (5.108)$$

The term  $\overline{Q}_N$  follows directly from Eq ( 4.55):

$$\overline{Q}_N = 2W \quad (5.109)$$

Combining the two equations above yields the following expression:

$$\overline{Q}_{S+N} - \overline{Q}_N = S^2 \quad (5.110)$$

The above equation is reduced by a factor of  $b\tau$  to account for the narrow video bandwidth. This results in

$$\overline{Q}_{S+N} - \overline{Q}_N = S^2 b\tau \quad (5.111)$$

Without CW interference present, the denominator of the expression of the deflection criterion,  $[\overline{Q}_N^2 - (\overline{Q}_N)^2]^{\frac{1}{2}}$  is derived from the PSD for the case of noise only,  $S_Q(f)$ . First, the autocorrelation function is obtained by the application of the expression for  $\overline{Q_1 Q_2}$ , Eq ( 4.66) of Section 4.2.1. In the noise only case  $S = 0$  and  $\alpha = \beta = 0$ . Therefore

$$\overline{Q_1 Q_2} = R_Q(\tau) = 4W^2 \rho^2(\tau) \quad (5.112)$$

The PSD is derived by taking the Fourier cosine transform of the autocorrelation function:

$$\begin{aligned}
S_Q(f) &= 2 \int_0^\infty R_Q(\tau) \cos \omega \tau d\tau \\
&= 8W^2 \int_0^\infty \rho^2(\tau) \cos \omega \tau d\tau \\
&= 8W^2 \int_0^\infty \exp \left[ - \left( \frac{\pi \tau B}{a} \right)^2 \right] \cos \omega \tau d\tau
\end{aligned} \tag{5.113}$$

The integral in the above equation was solved in the previous section by the use of Eq (679) from the *CRC Tables* (16:466). It reduces to

$$S_Q(f) = \frac{4\sqrt{2}a}{B\sqrt{\pi}} \exp \left( - \frac{a^2 f^2}{B^2} \right) \tag{5.114}$$

The variance,  $\overline{Q_N^2} - (\overline{Q_N})^2$ , is merely  $2S_Q(0)b$ :

$$\overline{Q_N^2} - (\overline{Q_N})^2 = \frac{8W^2 ab}{B\sqrt{\pi}} \tag{5.115}$$

Substituting Eqs ( 5.111) and ( 5.115) into the expression for the deflection criterion, Eq ( 5.29), one obtains

$$S^2 b \tau = C \left( \frac{8W^2 ab}{B\sqrt{\pi}} \right)^{\frac{1}{2}} \tag{5.116}$$

Similar to the derivation in the previous section, the above equation reduces to the following equation for  $\overline{P}_{\min}(\infty)$ :

$$\overline{P}_{\min}(\infty) = \frac{2\sigma^2 C}{T_o} \sqrt{\frac{\pi B}{2ab}} \left( \frac{4}{\pi} \right)^{\frac{1}{4}} \tag{5.117}$$

To compare on-frequency CW jamming with the case of large detuning one obtains



$$\begin{aligned}\frac{\overline{P}_{\min}(0)}{\overline{P}_{\min}(\infty)} &= \frac{1}{E} \left( \frac{\pi^2}{2} \right)^{\frac{1}{4}} \\ &= \frac{1}{E} (1.49)\end{aligned}\tag{5.118}$$

This value for the on-frequency jamming is contrasted with the value of 4.75 dB obtained for linear detection. Here, the magnitude of the CW jamming signal,  $E$ , has a direct impact on the minimum power required.

Although no experimental data was available for the quadratic detector, theoretical values of  $\overline{P}_{\min}/\overline{P}_{\min}(0)$  are derived for values of  $\tau|f_{cw} - f_s| \leq 2.5$ . These values are tabulated in Table 5.2. As before,  $B\tau = 2.2$ . For values of  $\tau|f_{cw} - f_s| > 2.5$ , the large detuning causes the effects of the CW interference to be negligible. This follows from the experimental data obtained for the linear detector. The theoretical values in Table 5.2 are calculated from

$$\frac{\overline{P}_{\min}}{\overline{P}_{\min}(0)} = \left[ \frac{\pi(f_{cw} - f_s)\tau}{\sin \pi(f_{cw} - f_s)\tau} \right]^2\tag{5.119}$$

Table 5.2. Theoretical Values of Signal Threshold Due to CW Detuning for a Quadratic Detector

$\tau f_{cw} - f_s $	$\overline{P}_{\min}/\overline{P}_{\min}(0)$ , dB
0.5	3.9
1	$\infty$
1.5	13.5
2	$\infty$
2.5	17.9

## VI. Conclusions and Recommendations

### 6.1 Conclusions

From the expression for the minimum power required for signal detection by the linear detector, Eq ( 5.63), the minimum power is a function of the frequency difference  $f_{cw} - f_s$ . For on-frequency interference,  $f_{cw} = f_s$ , Lawson and Uhlenbeck show from experiment that the minimum required signal power is 3.5 dB higher than that required when no CW interference is present (11:351-352). This increase in required power is compared with the theoretical value of 4.75 dB (11:352). The theoretical and experimental data in Table 5.1 show that the minimum power required for detection increases as the detuning increases until the detuning is so great as to render the CW interference negligible (11:352).

Similarly, the minimum required power for the quadratic detector is given by Eq ( 5.106). This equation shows that the minimum power for the quadratic detector is a function of both the frequency difference,  $f_{cw} - f_s$ , and the magnitude of the CW signal,  $E$ . The minimum power required when the interference is on-frequency versus the power required without CW interference is  $\frac{1}{E}(1.49)$ . The data tabulated in Table 5.2 show that as is the case for linear detection, the minimum power required increases until the detuning is so great that the effects of the CW interference are negligible.

The effect of a pure CW tone as an interfering signal is not a significant degradation. The effect of CW interference is especially limited when the frequency of the CW signal differs greatly from that of the carrier frequency of the pulsed radar signal. However, when some other modulation is present with the CW interference, the combined effects should be much greater than that of an unmodulated CW signal.

## 6.2 Recommendations

It is recommended that follow-on work be accomplished on this study of the effects of noise jamming on radar receivers. This follow-on work would include applying the analysis techniques presented here to the effects of CW interference on different pulsed radar receiver types. Specifically, these receivers would include matched filter, pulse compression and CFAR receivers. Additionally, various jamming waveforms should be substituted for the CW interference, and the effects on the various receivers should be analyzed. Lastly, instead of the application of the deflection criterion, an analysis by the use of a likelihood ratio test should be used to determine the effects of the different jamming waveforms on the radar receiver's  $P_d$  and  $P_{fa}$ .

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